DOI：10.13718/j.cnki.xdzk.2017.08.013

含两个非线性项的 Gronwall-Bellman 型非连续函数积分不等式的推广[①]

李自尊，　柳长青

百色学院 数学与统计学院，广西 百色 533000

摘要：研究了含有未知函数的两个非线性项的非连续函数积分不等式，利用分析技巧给出了未知函数的上界估计，并利用此结果估计了脉冲微分方程的上界.

关　键　词：非连续函数积分不等式；未知函数估计；脉冲微分系统

中图分类号：O178　　**文献标志码**：A　　**文章编号**：1673-9868(2017)08-0089-08

积分不等式是研究微分方程和积分方程的重要工具. 通过对积分不等式中未知函数的估计，可以研究某些微分方程解的存在性、有界性、唯一性和稳定性等定性性质[1-17]. 通过对非连续函数积分不等式中未知函数进行估计，可以研究某些脉冲微分方程和解的一些性质.

文献[3]研究了积分不等式

$$u(t) \leqslant \varphi(t) + \int_{t_0}^{t} g(s) u^m(s) \mathrm{d}s + \sum_{t_0 < t_i < t} \beta_i u^m(t_i - 0) \qquad \forall\, t \geqslant t_0$$

文献[7]研究了下面的非连续函数积分不等式

$$u(t) \leqslant a(t) + q(t) \left[\int_{t_0}^{t} f(s) u(\tau(s)) \mathrm{d}s + \int_{t_0}^{t} f(s) \left(\int_{t_0}^{s} g(t) u(\tau(t)) \mathrm{d}t \right) \mathrm{d}s + \sum_{t_0 < t_i < t} \beta_i u^m(t_i - 0) \right] \qquad \forall\, t \geqslant t_0$$

其中，$a(t) > 0$，$q(t) \geqslant 1$，$f(t) \geqslant 0$，$g(t) \geqslant 0$，$\beta_i \geqslant 0$.

文献[16]研究了含有时滞的脉冲积分不等式

$$u(t) \leqslant a(t) + \int_{t_0}^{t} f(t, s) u(\tau(s)) \mathrm{d}s + \int_{t_0}^{t} f(t, s) \left(\int_{t_0}^{s} g(s, \theta) u(\tau(\theta)) \mathrm{d}\theta \right) \mathrm{d}s + q(t) \sum_{t_0 < t_i < t} \beta_i u^m(t_i - 0)] \qquad \forall\, t \geqslant t_0$$

文献[12]研究了含有未知函数的复合函数的积分不等式

$$u(t) \leqslant a(t) + \int_{t_0}^{t} f(t, s) \int_{t_0}^{s} g(s, \tau) w(u(\tau)) \mathrm{d}\tau \mathrm{d}s +$$

---

① 收稿日期：2016-11-04

基金项目：国家自然科学基金项目(11561019)；广西自然科学基金项目(2013GXNSFAA019022)；广西教育厅项目(201204LX423，2013LX148，KY2015YB280).

作者简介：李自尊(1984-)，男，山东济宁人，讲师，主要从事微分方程与动力系统的研究.

通信作者：柳长青，副教授.

$$q(t)\sum_{t_0<t_i<t}\beta_i u^m(t_i-0)] \qquad \forall t\geqslant t_0$$

这里 $w(u)$ 是定义在 $[0,\infty)$ 上的单调不减连续函数且当 $u>0$ 时，$w(u)>0$. 本文在上述研究成果的基础上，研究了一类含三项未知函数复合的非连续函数积分不等式

$$\phi(u(t))\leqslant a(t)+\int_{t_0}^{t}f_1(t,s)w_1(u(s))\mathrm{d}s+$$

$$\int_{t_0}^{t}f(t,s)\left(\int_{t_0}^{s}g(s,\tau)w_2(u(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\sum_{t_0<t_i<t}\beta_i\phi(u(t_i-0)) \tag{1}$$

其中，$u(t)$ 定义在是 $[t_0,\infty)$ 上的只有第一类不连续点 $\{t_i: t_0<t_1<t_2\cdots,\lim\limits_{i\to\infty}t_i=\infty\}$ 的非负逐段连续函数，$\phi(u)$ 是定义在 $[0,\infty)$ 上的正的严格单调递增函数，$m>1$，$\beta_i\geqslant 0$，m，β_i 是给定的常数.

1 主要结论

假设

(H_1) ϕ 在 $[0,\infty)$ 是严格增的连续函数，对任意的 $u>0$，$\psi(u)>0$；

(H_2) $w_i(i=1,2)$ 在 $[0,\infty)$ 上是连续不减函数，在 $(0,\infty)$ 上是正的，且 $\dfrac{w_2}{w_1}$ 是不减的；

(H_3) $a(t)$ 是定义在 $[t_0,\infty)$ 上的连续函数，$a(t_0)\neq 0$；

(H_4) $f_i(t,s)$ $(i=1,2)$ 和 $f(t,s)$，$g(s,t)$ 是定义在 $[t_0,\infty)\times[t_0,\infty)$ 上的非负连续函数；

(H_5) $\beta_i\geqslant 0$ 是常数.

定理1 具有第一类不连续点 $\{t_i: t_0<t_1<t_2\cdots,\lim\limits_{i\to\infty}t_i=\infty\}$ 的非负逐段连续函数 $u(t)$ $(t\geqslant t_0\geqslant 0)$ 满足积分不等式(1)，则函数 $u(t)$ 有下面的估计式：

$$u(t)\leqslant\phi^{-1}(W_2^{-1}(e_3(t))) \qquad \forall t\in[t_i,t_{i+1})$$

其中

$$W_i(u)=\int_1^u\frac{\mathrm{d}s}{w_i(\phi^{-1}(s))} \qquad i=1,2$$

$$\widetilde{f}_i(t,s)=\max_{t_0\leqslant\tau\leqslant t}f_i(\tau,s) \qquad i=1,2$$

$$e_1(t)=\max_{t_0\leqslant\tau\leqslant t}|a(\tau)|$$

$$E_i(t)=e_1(t)+\sum_{k=0}^{i-1}\sum_{j=1}^{2}\int_{t_k}^{t_{k+1}}\widetilde{f}_j(t,s)w_j(u(s))\mathrm{d}s+$$

$$\beta_k(\phi(u(t_k-0))) \qquad \forall t\in[t_i,t_{i+1}) \qquad i=1,2,\cdots$$

$$e_2=W_1(E_1(t))+\int_{t_0}^{t}\widetilde{f}_1(t,s)\mathrm{d}s \qquad \forall t\in[t_0,t_1)$$

$$e_3=W_2(W_1^{-1}(e_2(t)))+\int_{t_0}^{t}\widetilde{f}_2(t,s)\mathrm{d}s \qquad \forall t\in[t_0,t_1)$$

$$e_2=W_1(E_i(t))+\int_{t_i}^{t}\widetilde{f}_1(t,s)\mathrm{d}s \qquad \forall t\in[t_i,t_{i+1}) \qquad i=1,2,\cdots$$

$$e_3=W_2(W_1^{-1}(e_2(t)))+\int_{t_i}^{t}\widetilde{f}_2(t,s)\mathrm{d}s \qquad \forall t\in[t_i,t_{i+1}) \qquad i=1,2,\cdots \tag{2}$$

证 令

$$\widetilde{f}_2(t,s)=\int_s^t\max_{t_0\leqslant\tau\leqslant t}f(t,\tau)g(\tau,s)\mathrm{d}\tau \tag{3}$$

由于 $f(t,s)$，$g(t,s)$，$w(u(t))$ 都是连续函数，得

$$\int_{t_0}^{t} f(t,s)\int_{t_0}^{s} g(s,\tau)w(u(\tau))\mathrm{d}\tau\mathrm{d}s=$$
$$\int_{t_0}^{t} w(u(\tau))\int_{\tau}^{t} f(t,s)g(s,\tau)\mathrm{d}s\ \mathrm{d}\tau=$$
$$\int_{t_0}^{t} w(u(s))\int_{\tau}^{t} f(t,\tau)g(\tau,s)\mathrm{d}\tau\mathrm{d}s\leqslant$$
$$\int_{t_0}^{t} \widetilde{f}_2(t,s)w(u(s))\mathrm{d}s \tag{4}$$

由(2),(4)，则(1) 式变为

$$\phi(u(t))\leqslant e_1(t)+\sum_{i=1}^{2}\int_{t_0}^{t}\widetilde{f}_i(t,s)w_i(u(s))\mathrm{d}s+\sum_{t_0<t_i<t}\beta_i\phi(u(t_i-0)) \tag{5}$$

首先，我们考虑情况 $t\in[t_0,t_1)$，任取 $T\in[t_0,t_1)$，对任意 $t\in[t_0,T]$，由(5) 式，可得

$$\phi(u(t))\leqslant e_1(t)+\sum_{i=1}^{2}\int_{t_0}^{t}\widetilde{f}_i(T,s)w_i(u(s))\mathrm{d}s \tag{6}$$

令

$$v(t)=e_1(t)+\sum_{i=1}^{2}\int_{t_0}^{t}\widetilde{f}_i(T,s)w_i(u(s))\mathrm{d}s \tag{7}$$

则 $v(x)$ 为非负不减的连续函数，且

$$\phi(u(t))\leqslant v(t)\qquad u(t)\leqslant \phi^{-1}(v(t))\qquad v(t_0)=e_1(t_0) \tag{8}$$

对式(7) 求导，得

$$v'(t)=e_1'(t)+\sum_{i=1}^{2}\widetilde{f}_i(T,t)w_i(u(t)) \tag{9}$$

令

$$\psi_i(t)=\frac{w_i(t)}{w_1(t)}\qquad i=1,2 \tag{10}$$

由(9) 和(10) 式可得

$$\frac{v'(t)}{w_1(\phi^{-1}(v(t)))}=\frac{e_1'(t)+\sum_{i=1}^{2}\widetilde{f}_i(T,t)w_i(u(t))}{w_1(\phi^{-1}(v(t)))}\leqslant$$
$$\frac{e_1'(t)+\sum_{i=1}^{2}\widetilde{f}_i(T,t)w_i(\phi^{-1}(v(t)))}{w_1(\phi^{-1}(v(t)))}=$$
$$\frac{e_1'(t)}{w_1(\phi^{-1}(v(t)))}+f_1(T,t)+\frac{\widetilde{f}_2(t,t)w_2(\phi^{-1}(v(t)))}{w_1(\phi^{-1}(v(t)))}=$$
$$\frac{e_1'(t)}{w_1(\phi^{-1}(v(t)))}+f_1(T,t)+\widetilde{f}_2(T,t)\psi_2(\phi^{-1}(v(t))) \tag{11}$$

对(10) 式两边从 t_0 到 t 同时积分，并利用 $W_i(t)$ 的定义，我们得到

$$W_1(v(t))-W_1(v(t_0))\leqslant W_1(e_1(t))-W_1(e_1(t_0))+\int_{t_0}^{t}\widetilde{f}_1(T,s)\mathrm{d}s+$$
$$\int_{t_0}^{t}\widetilde{f}_2(T,s)\psi_2(\phi^{-1}(v(s)))\mathrm{d}s \tag{12}$$

由于 $W_1(v(t_0))=W_1(e_1(t_0))$，则(11) 式可写为

$$W_1(v(t)) \leqslant W_1(e_1(t)) + \int_{t_0}^{t} \widetilde{f}_1(T,s)\mathrm{d}s + \int_{t_0}^{t} \widetilde{f}_2(T,s)\psi_2(\phi^{-1}(v(s)))\mathrm{d}s \tag{13}$$

令

$$\theta_1(t) = W_1(v(t)) \tag{14}$$

$$e_2(t) = W_1(e_1(t)) + \int_{t_0}^{t} \widetilde{f}_1(T,s)\mathrm{d}s \tag{15}$$

由(14)和(15),则(13)式变为

$$\begin{aligned}\theta_1(t) &\leqslant e_2(t) + \int_{t_0}^{t} \widetilde{f}_2(T,s)\psi_2(\phi^{-1}(v(s)))\mathrm{d}s \leqslant \\ &e_2(t) + \int_{t_0}^{t} \widetilde{f}_2(T,s)\psi_2(\phi^{-1}(W_1^{-1}(\theta_1(s))))\mathrm{d}s\end{aligned} \tag{16}$$

令

$$v_1(t) = e_2(t) + \int_{t_0}^{t} \widetilde{f}_2(T,s)\psi_2(\phi^{-1}(W_1^{-1}(\theta_1(s))))$$

则 $v_1(t)$ 在 $[t_0, t_1)$ 是连续不减的函数,且

$$\theta_1(t) \leqslant v_1(t) \qquad v_1(t_0) = e_2(t_0)$$

定义函数

$$\Phi_2(u) = \int_0^u \frac{\mathrm{d}s}{\psi_2(\phi^{-1}(W_1^{-1}(s)))} \tag{17}$$

则

$$\begin{aligned}\frac{v_1'(t)}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))} &= \frac{e_2'(t) + \widetilde{f}_2(T,t)\psi_2(\phi^{-1}(W_1^{-1}(\theta_1(t))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))} \leqslant \\ &\frac{e_2'(t) + \widetilde{f}_2(T,t)\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))}\end{aligned} \tag{18}$$

对(18)式的两边,从 t_0 到 t 积分,我们得到

$$\begin{aligned}&\int_{t_0}^{t} \frac{v_1'(s)\mathrm{d}s}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))} \leqslant \\ &\int_{t_0}^{t} \frac{e_2'(s) + \widetilde{f}_2(T,s)\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))}\mathrm{d}s \leqslant \\ &\int_{t_0}^{t} \frac{e_2'(s)\mathrm{d}s}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} + \int_{t_0}^{t} \widetilde{f}_2(T,s)\mathrm{d}s\end{aligned} \tag{19}$$

由(10),(17),(19)式可得

$$\begin{aligned}\Phi_2(v_1(t)) - \Phi_2(v_1(t_0)) &\leqslant \int_{t_0}^{t} \frac{e_2'(s)\mathrm{d}s}{\psi_2(\phi^{-1}(W_1^{-1}(e_2(s))))} + \int_{t_0}^{t} \widetilde{f}_2(T,s)\mathrm{d}s \leqslant \\ &\Phi_2(e_2(t)) - \Phi_2(e_2(t_0)) + \int_{t_0}^{t} \widetilde{f}_2(T,s)\mathrm{d}s\end{aligned} \tag{20}$$

由(20)式可得

$$\Phi_2(v_1(t)) \leqslant \Phi_2(e_2(t)) + \int_{t_0}^{t} \widetilde{f}_2(T,s)\mathrm{d}s \tag{21}$$

由(17)式,我们可以推出

$$\Phi_2(u) = \int_0^u \frac{\mathrm{d}s}{\psi_2(\phi^{-1}(W_1^{-1}(s)))} =$$

$$\int_0^u \frac{w_1(\phi^{-1}(W_1^{-1}(s)))\,\mathrm{d}s}{w_2(\phi^{-1}(W_1^{-1}(s)))} = \int_1^{W_1^{-1}(u)} \frac{\mathrm{d}s}{w_2(\phi^{-1}((s)))} = W_2(W_1^{-1}(u)) \tag{22}$$

由(22),(21) 式可变为

$$W_2(W_1^{-1}(v_1(t))) \leqslant W_2(W_1^{-1}(e_2(t))) + \int_{t_0}^t \widetilde{f}_2(T, s)\mathrm{d}s \tag{23}$$

由(23) 式可推出

$$v_1(t) \leqslant W_1\left(W_2{}^{-1}(W_2(W_1^{-1}(e_2(t))) + \int_{t_0}^t \widetilde{f}_2(T, s)\mathrm{d}s)\right) \tag{24}$$

由(8)，(14) 和(24) 式可得

$$u(t) \leqslant \phi^{-1}\left(W_2{}^{-1}(W_2(W_1^{-1}(e_2(t))) + \int_{t_0}^t \widetilde{f}_2(T, s)\mathrm{d}s)\right) \leqslant \phi^{-1}(W_2{}^{-1}(e_3(t))) \qquad t \in [t_0, T]$$

其中

$$e_3(t) = W_2(W_1^{-1}(e_2(t))) + \int_{t_0}^t \widetilde{f}_2(T, s)\mathrm{d}s \qquad t \in [t_0, T)$$

由 T 的任意性可得

$$u(t) \leqslant \phi^{-1}\left(W_2{}^{-1}(W_2(W_1^{-1}(e_2(t))) + \int_{t_0}^t \widetilde{f}_2(t, s)\mathrm{d}s)\right) \leqslant \phi^{-1}(W_2{}^{-1}(e_3(t))) \qquad t \in [t_0, t_1)$$

当 $t \in [t_0, t_1)$ 时我们证明了估计式.

当 $t \in [t_1, t_2)$ 时，任意确定 $T_1 \in [t_1, t_2)$，对于任意的 $t \in [t_1, T_1]$，不等式(4) 变为

$$\begin{aligned}\phi(u(t)) &\leqslant e_1(t) + \sum_{i=1}^2 \int_{t_0}^t \widetilde{f}_i(T_1, s)w_i(u(s))\mathrm{d}s + \beta_1\phi(u(t_1-0)) \leqslant \\ &e_1(t) + \sum_{i=1}^2 \int_{t_0}^{t_1} \widetilde{f}_i(T_1, s)w_i(u(s))\mathrm{d}s + \\ &\sum_{i=1}^2 \int_{t_1}^t \widetilde{f}_i(T_1, s)w_i(u(s))\mathrm{d}s + \beta_1\phi(u(t_1-0))\end{aligned} \tag{25}$$

令 $\Gamma(t)$ 表示(25) 式的右边，

$$E_1(t) = e_1(t) + \sum_{i=1}^2 \int_{t_0}^{t_1} \widetilde{f}_i(t, s)w_i(u(s))\mathrm{d}s + \beta_1\phi(u(t_1-0))$$

则 $\Gamma(t)$ 是单调不减函数，且有

$$\phi(u(t)) \leqslant \Gamma(t)$$

$$\phi(u(t_1)) \leqslant \Gamma(t_1) = E_1(t_1) = e_1(t_1) + \sum_{i=1}^2 \int_{t_0}^{t_1} \widetilde{f}_i(T_1, s)w_i(u(s))\mathrm{d}s + \beta_1\phi(u(t_1-0)) \tag{26}$$

对 $\Gamma(t)$ 的两边关于 t 求导得

$$\Gamma'(t) \leqslant E_1'(t) + \sum_{i=1}^2 \widetilde{f}_i(T_1, t)w_i(u(t)) \leqslant E_1'(t) + \sum_{i=1}^2 \widetilde{f}_i(T_1, t)w_i(\phi^{-1}(\Gamma(t))) \tag{27}$$

使(27)式两边同时除以 $w_1(\phi^{-1}(\Gamma(t)))$，可得

$$\frac{\Gamma'(t)}{w_1(\phi^{-1}(\Gamma(t)))} \leqslant \frac{E_1'(t) + \sum_{i=1}^{2} \widetilde{f}_i(T_1, t) w_i(\phi^{-1}(\Gamma(t)))}{w_1(\phi^{-1}(\Gamma(t)))} \tag{28}$$

又对(28)式两边从 t_1 到 t 积分可得

$$W_1(\Gamma(t)) - W_1(\Gamma(t_1)) \leqslant \int_{t_1}^{t} \frac{E_1'(s) + \sum_{i=1}^{2} \widetilde{f}_i(T_1, s) w_i(\phi^{-1}(\Gamma(s)))}{w_1(\phi^{-1}(\Gamma(s)))} \mathrm{d}s \leqslant$$

$$W_1(E_1(t)) - W_1(E_1(t_1)) + \int_{t_1}^{t} \widetilde{f}_1(T_1, s)\mathrm{d}s +$$

$$\int_{t_1}^{t} \widetilde{f}_2(T_1, s)\psi_2(\phi^{-1}(\Gamma(s)))\mathrm{d}s$$

则

$$W_1(\Gamma(t)) \leqslant W_1(E_1(t)) + \int_{t_1}^{t} \widetilde{f}_1(T_1, s)\mathrm{d}s + \int_{t_1}^{t} \widetilde{f}_2(T_1, s)\psi_2(\phi^{-1}(\Gamma(s)))\mathrm{d}s \tag{29}$$

从而(28)式变为了(11)式的形式，利用相同的方法可以得到估计式

$$u(t) \leqslant \phi^{-1}(W_2^{-1}(e_3(t))) \qquad \forall t \in [t_1, t)$$

同理，对任意自然数 k，当 $t \in [t_k, t_{k+1})$ 时，我们可以得到未知函数的估计式

$$u(t) \leqslant \phi^{-1}(W_2^{-1}(e_3(t))) \qquad \forall t \in [t_k, t_{k+1})$$

综上定理被证明.

2 在脉冲微分方程中的应用

本节我们用得到的结果给出脉冲微分系统解的上界估计. 考虑脉冲微分系统

$$\frac{\mathrm{d}(x(t))}{\mathrm{d}t} = F(t, x) \qquad t \neq t_i,\ t \in [t_0, \infty) \tag{30}$$

$$\Delta(x)\big|_{t=t_i} = \beta_i x(t_i - 0) \tag{31}$$

$$x(t_0) = c$$

其中：$0 \leqslant t_0 < t_1 < t_2 < \cdots$，$\lim\limits_{i\to\infty} t_i = \infty$，$c > 1$ 是常数，$F(t, x)$ 关于 t，x 在$[t_0, \infty) \times s(-\infty, +\infty)$ 上连续. 假设(30)式中 $F(t, x)$ 满足

$$|F(t, x)| \leqslant f_1(t)|x^{\frac{1}{2}}| + f_2(x)e^{|x|} \tag{32}$$

其中 $f_1(t)$，$f_2(t)$ 是$[t_0, \infty)$ 上连续的非负函数.

推论 1 在条件(32)式成立的情况下，系统(30)，(31)式所有的解 $x(t)$ 满足估计式：

$$u(t) \leqslant W_2^{-1}(e_3(t)) \qquad \forall t \in [t_i, t_{i+1}) \tag{33}$$

其中

$$W_1(u) = \int_0^u \frac{\mathrm{d}s}{s^{\frac{1}{2}}} = 2u^{\frac{1}{2}} \qquad W_1^{-1}(u) = \frac{u^2}{4}$$

$$W_2(u) = \int_0^u \frac{\mathrm{d}s}{e^s} = 1 - e^{-u} \qquad W_2^{-1}(u) = -\ln(1-u)$$

$$\widetilde{f}_1(t, s) = f_1(s) \qquad \widetilde{f}_2(t, s) = f_2(s)$$

$$e_1(t) = c$$

$$E_i(t)=c+\sum_{k=0}^{i}\sum_{j=1}^{2}\int_{t_{k-1}}^{t_k} f_j(s)w_j(u(s))\mathrm{d}s+$$

$$\beta_k(\phi(u(t_i-1)))\qquad \forall t\in[t_i,t_{i+1})\qquad i=1,2,\cdots$$

$$e_2=W_1(e_1(t))+\int_{t_0}^{t} f_1(s)\mathrm{d}s\qquad \forall t\in[t_0,t_1)$$

$$e_3=W_2(W_1^{-1}(e_2(t)))+\int_{t_0}^{t} f_2(s)\mathrm{d}s\qquad \forall t\in[t_0,t_1)$$

$$e_2=W_1(E_i(t))+\int_{t_i}^{t} f_1(s)\mathrm{d}s\qquad \forall t\in[t_i,t_{i+1})\qquad i=1,2,\cdots$$

$$e_3=W_2(W_1^{-1}(e_2(t)))+\int_{t_i}^{t} f_2(s)\mathrm{d}s\qquad \forall t\in[t_i,t_{i+1})$$

证　脉冲微分方程(30)与(31)式等价于积分方程

$$x(t)=c+\int_{t_0}^{t} F(s,x(s))\mathrm{d}s+\sum_{t_0<t_i<t}\beta_i x(t_i-0),\ t\in[t_0,\infty)\tag{34}$$

利用条件(32)，由(34)式，可得

$$|x(t)|\leqslant c+\int_{t_0}^{t} f_1(s)|x^{\frac{1}{2}}(s)|\mathrm{d}s+\int_{t_0}^{t} f_2(s)e^{|x(s)|}\mathrm{d}s\sum_{t_0<t_i<t}\beta_i|x(t_i-0)|\tag{35}$$

令 $u(t)=|x(t)|$，由(35)式，我们可得不等式

$$u(t)\leqslant c+\int_{t_0}^{t} f_1(s)u^{\frac{1}{2}}(s)\mathrm{d}s+\int_{t_0}^{t} f_2(s)\ e^{u(s)}+\sum_{t_0<t_i<t}\beta_i u(t_i-0)\tag{36}$$

令

$$w_1(u)=u^{\frac{1}{2}}\qquad w_2(u)=e^u$$

我们看出(36)式是(5)式的特殊形式. 且(36)式中的函数满足定理1的条件，由定理1，我们可以推出 $x(t)$ 的估计式(33)式.

参考文献：

[1] AGARWAL R P, DENG Sheng-fu, ZHANG Wei-nian. Generalization of a Retarded Gronwall-Like Inequality and Its Applications [J]. Appl Math Comput, 2005, 165: 599－612.

[2] BORYSENKO S D. About Asymptotical Stability on Linear Approximation of the Systems with Impulse Influence [J]. Ukrain Mat Zh, 1983, 35(2): 144－150.

[3] BORYSENKO S D. About One Integral Inequality for Piece-Wise Continuous Functions [M]. Proc. XInt: Kravchuk Conf, Kyiv, 2004: 323.

[4] BORYSENKO S D, CIARLETTA M, IOVANE G. Integro-Sum Inequalities and Motion Stability of Systems with Impulse Perturbations [J]. Nonlnear Anal, 2005, 62: 417－428.

[5] GLLO A, PICCIRILO A M. About New Analogies of Gronwall-Bellman-Bihari Type Inequalities for Discontinuous Functions and Estimated Solutions for Impulsive Differential Systems [J]. Nonlnear Anal, 2007, 67: 1550－1559.

[6] GLLO A, PICCIRILO A M. About Some New Generalizations of Bellman-Bihari Results for Integro-Functional Inequalities with Discontinuous Functions and Applications [J]. Nonlnear Anal, 2009, 71: e2276－e2287.

[7] IOVANE G. Some New Integral Inequalities of Bellman-Bihari Type with Delay for Discontinuous Functions [J]. Nonlnear Anal, 2007, 66: 498－508.

[8] 李自尊. 脉冲积分不等式未知函数的估计 [J]. 四川师范大学学报(自然科学版), 2013, 36(2): 258－262.

[9] 柳长青，李自尊. 一类新的非连续函数积分不等式及其应用 [J]. 理论数学, 2013(3): 4－8.

[10] 孟东沅. 一类新型不连续函数的积分不等式及应用 [J]. 数学的实践与认识, 2009, 39: 161－166.

[11] MI Yu-zhen, DENG Sheng-fu, LI Xiao-pei. Nonlinear Integral Inequalities with Delay for Discontinuous Functions and Their Applications [J]. J Inequal Appl, 2013, 430: 11.

[12] 米玉珍,钟吉玉. 非连续函数的 Bellman-Bihari 型积分不等式的推广 [J]. 四川大学学报(自然科学版), 2015, 52(1): 33-38.

[13] MITROPOLSKIY YU A, IOVANE G, BORYSENKO S D. About a Generalization of Bellman-Bihari Type Inequalities for Discontinuous Functions and Their Applications [J]. Nonlnear Anal, 2007, 66: 2140-2165.

[14] WANG Wu-sheng. A Generalized Retarded Gronwall-Like Inequality in Two Variables and Applications to BVP [J]. Appl Math Comput, 2007, 191(1): 144-154.

[15] WANG Wu-sheng, LI Zi-zun. A New Class of Impulsive Integral Inequalities and Its Application [C]. 2011 International Conference on Multimedia Technology (ICMT 2011), 2011: 1897-1899.

[16] 严 勇. 一类带脉冲项的 Gronwall-Bellman 型积分不等式的推广及应用 [J]. 四川师范大学学报(自然科学版), 2013, 36(4): 603-609.

[17] ZHENG Zhao-wen, GAO Xin, SHAO Jing. Some New Generalized Retarded Inequalities for Discontinuous Functions and Their Applications [J]. J Inequal Appl, 2016(7): 14.

Generalization of a Class of Integral Inequalities with Gronwall-Bellman Type for Discontinuous Functions

LI Zi-zun, LIU Chang-qing

School of Mathematics, Baise University, Baise Guangxi 533000, China

Abstract: In this paper, we give the upper bound estimation of an unknown function containing three nonlinear terms of integral inequality for discontinuous functions. The result is used to estimate the upper bounds of impulsive differential equations.

Key words: integral inequality for discontinuous functions; estimation of unknown functions; impulsive differential system

责任编辑 周仁惠
崔玉洁