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朝代循环模型的分支方向及稳定性①

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摘要： 对于具有时滞的朝代循环模型，当时间延迟到达或穿过临界值时，系统在正平衡点附近出现了一族周期解. 应用规范型和中心流形理论给出决定该模型分支方向及分支周期解稳定性的显示表达式.

关　键　词： Hopf 分支；稳定性；周期解

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文献[1] 研究了一个具有 3 个种群的社会模型，模型中 3 个种群分别为农民(x)、强盗(y) 和统治者(z). 文献[2] 在此系统中引入了时间延迟 τ，证明了时滞系统

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x\left[1-x(t-\tau)-\dfrac{y}{B+x}-Hz\right]=x\varphi(x,y,z)\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=Qy\left(\dfrac{Ex}{B+x}-1-\dfrac{z}{D+y}\right)=Qy\psi(x,y,z)\\ \dfrac{\mathrm{d}z}{\mathrm{d}t}=R\left(\dfrac{xy}{B+x}-Gz\right)=R\varphi(x,y,z)\end{cases}\tag{1}$$

当时间延迟 τ 到达或穿过临界值 $\tau^{(j)}$(其中 $\tau^{(j)}$ 如文献[1] 中所定义)时，在正平衡点附近出现了一族周期解. 这里我们假设当 $\tau=\tau^{(j)}$ 时系统(1) 在正平衡点处出现了 Hopf 分支，下面应用标准形理论和中心流形理论来研究分支周期解的稳定性[1-7].

令系统(1) 的正平衡点为 x^*,y^*,z^*. 令 $u_1(t)=x(t)-x^*$，$u_2(t)=y(t)-y^*$，$u_3(t)=z(t)-z^*$，$x_i(t)=u_i(\tau t)$，$\tau=\tau^{(j)}+\mu$ ($\mu\in\mathbb{R}$). 系统(1) 等价于系统

$$\begin{cases}\dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t}=(\tau+\mu)(x^*+u_1(t))\left[\dfrac{y^*}{B+x^*}-u_1(t-1)-\dfrac{y^*+u_2(t)}{B+x^*+u_1(t)}-Hu_3(t)\right]\\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t}=(\tau+\mu)Q(y^*+u_2(t))\left[E\dfrac{x^*+u_1(t)}{B+x^*+u_1(t)}-\dfrac{Ex^*}{B+x^*}+\dfrac{z^*}{D+y^*}-\dfrac{z^*+u_3(t)}{D+y^*+u_2(t)}\right]\\ \dfrac{\mathrm{d}u_3(t)}{\mathrm{d}t}=(\tau+\mu)R\left[\dfrac{(x^*+u_1(t))(y^*+u_2(t))}{B+x^*+u_1(t)}-\dfrac{x^*y^*}{B+x^*}-Gu_3(t)\right]\end{cases}\tag{2}$$

此时系统(2) 具有平衡点(0，0，0).

定理 1　对系统(2) 应用规范型和中心流形理论给出了参数 g_{ij} 的计算公式，可以计算下列参数的值：

$$c_1(0)=\frac{\mathrm{i}}{2\tau^{(j)}\omega_0}\left(g_{11}g_{20}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{1}{2}g_{21}\qquad \kappa_2=-\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau^{(j)})\}}$$

$$\beta_2=2\mathrm{Re}\{c_1(0)\}\qquad T_2=-\frac{\mathrm{lm}\{c_1(0)\}+\kappa_2\mathrm{lm}\{\lambda'(\tau^{(j)})\}}{\tau^{(j)}\omega_0}$$

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这几个参数决定了系统当分支值变化到 $\tau^{(j)}$ 时的分支方向及分支周期解的相关性质. κ_2 决定了分支的方向：$\kappa_2>0$ 时，分支是上临界的，分之周期解在 $\tau>\tau^{(j)}$ 时出现；$\kappa_2<0$ 时，分支是下临界的，分支周期解在 $\tau<\tau^{(j)}$ 时出现. β_2 决定了分支周期解的稳定性：$\beta_2<0$ 时，中心流形上的周期解是稳定的；否则是不稳定的. T_2 决定了分支周期解的变化：$T_2>0$ 时，周期解的周期是增加的；否则是减少的.

证 选取相空间 $C=C([-1,0],\mathbb{R}^3)$，对 $\boldsymbol{\phi}(\theta)=(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta))^{\mathrm{T}}\in C$，令

$$\boldsymbol{L}_\mu\boldsymbol{\phi}=(\tau^{(j)}+\mu)\begin{pmatrix}\dfrac{x^*y^*}{(B+x^*)^2} & -\dfrac{x^*}{B+x^*} & -x^*H\\ \dfrac{EQBy^*}{(B+x^*)^2} & \dfrac{Qy^*z^*}{(D+y^*)^2} & \dfrac{Qy^*}{D+y^*}\\ \dfrac{RBy^*}{(B+x^*)^2} & \dfrac{Rx^*}{B+x^*} & -RG\end{pmatrix}\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\\ \phi_3(0)\end{pmatrix}+ (\tau^{(j)}+\mu)\begin{pmatrix}-x^* & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\\ \phi_3(-1)\end{pmatrix}\tag{3}$$

$$\boldsymbol{F}(\mu,\boldsymbol{\phi})=(\tau^{(j)}+\mu)\begin{pmatrix}\dfrac{B\phi_1(0)}{(B+x^*)^2}\left(1-\dfrac{\phi_1(0)}{B+x^*}\right)\left(\dfrac{y^*\phi_1(0)}{B+x^*}-\phi_2(0)-\phi_1(0)\varphi_1(1)-H\phi_1(0)\phi_3(0)\right)\\ \dfrac{QEB\phi_1(0)}{(B+x^*)^2}\left(\dfrac{\phi_1(0)}{B+x^*}-1\right)\left(\dfrac{y^*\phi_1(0)}{B+x^*}-\phi_2(0)\right)+\dfrac{QD\phi_2(0)}{(D+y^*)^2}\left(1-\dfrac{\phi_2(0)}{D+y^*}\right)\left(\dfrac{z^*\phi_2(0)}{D+y^*}-\phi_3(0)\right)\\ \dfrac{RBy^*\phi_1^2(0)}{(B+x^*)^3}\left(\dfrac{\phi_1(0)}{B+x^*}-1\right)+\dfrac{RB\phi_1(0)\phi_2(0)}{(B+x^*)^2}\left(1-\dfrac{\phi_1(0)}{B+x^*}\right)\end{pmatrix}\tag{4}$$

则系统(2)等价于

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=\boldsymbol{L}_\mu(x_t)+\boldsymbol{F}(\mu,x_t)\tag{5}$$

其中 $x_t(\theta)=x(t+\theta)$，$\theta\in[-1,0]$. 由文献[1]知，当 $\mu=0$ 时系统(2)的特征方程有一对纯虚根 $\pm\mathrm{i}\tau^{(j)}\omega_0$，且横截条件成立，系统在零平衡点处出现了 Hopf 分支. 由 Riesz 表示定理，存在着分量为有界变差函数的三阶矩阵 $\boldsymbol{\eta}(\theta,\mu)$，使得

$$\boldsymbol{L}_\mu\boldsymbol{\phi}=\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta,0)\boldsymbol{\phi}(\theta)\qquad \boldsymbol{\phi}\in C\tag{6}$$

事实上，只要取

$$\boldsymbol{\eta}(\theta,\mu)=(\tau^{(j)}+\mu)\begin{pmatrix}\dfrac{x^*y^*}{(B+x^*)^2} & -\dfrac{x^*}{B+x^*} & -x^*H\\ \dfrac{EQBy^*}{(B+x^*)^2} & \dfrac{Qy^*z^*}{(D+y^*)^2} & \dfrac{Qy^*}{D+y^*}\\ \dfrac{RBy^*}{(B+x^*)^2} & \dfrac{Rx^*}{B+x^*} & -RG\end{pmatrix}\boldsymbol{\delta}(\theta)+ (\tau^{(j)}+\mu)\begin{pmatrix}-x^* & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\boldsymbol{\delta}(\theta+1)$$

即可. 对 $\boldsymbol{\phi}\in C^1([-1,0],\mathbb{R}^3)$，定义

$$\boldsymbol{A}(\mu)\boldsymbol{\phi}=\begin{cases}\dfrac{\mathrm{d}\boldsymbol{\phi}(\theta)}{\mathrm{d}\theta} & \theta\in[-1,0)\\ \displaystyle\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\mu,s)\boldsymbol{\phi}(s) & \theta=0\end{cases}\qquad \boldsymbol{R}(\mu)\boldsymbol{\phi}=\begin{cases}0 & \theta\in[-1,0)\\ \boldsymbol{F}(\mu,\boldsymbol{\phi}) & \theta=0\end{cases}$$

这样方程(5)可写成如下形式

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=\boldsymbol{A}(\mu)x_t+\boldsymbol{R}(\mu)x_t \tag{7}$$

其中，对于任意 $\boldsymbol{\psi}\in C([0,1],(\mathbb{R}^3)^*)$，定义

$$\boldsymbol{A}^*\boldsymbol{\psi}(s)=\begin{cases}-\dfrac{\mathrm{d}\boldsymbol{\psi}(s)}{\mathrm{d}s} & s\in(0,1]\\ \displaystyle\int_{-1}^{0}\boldsymbol{\psi}(-t)\mathrm{d}\boldsymbol{\eta}(t,0) & s=0\end{cases}$$

和双线性内积

$$\langle\boldsymbol{\psi}(s),\boldsymbol{\phi}(\theta)\rangle=\overline{\boldsymbol{\psi}}(0)\boldsymbol{\phi}(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\boldsymbol{\psi}}(\xi-\theta)\mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{\phi}(\xi)\mathrm{d}\xi \tag{8}$$

其中 $\boldsymbol{\eta}(\theta)=\boldsymbol{\eta}(\theta,0)$，则 $\boldsymbol{A}(0)$ 与 $\boldsymbol{A}^*$ 互为共轭算子. 令 $\boldsymbol{q}(\theta)=(1,\alpha,\beta)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\theta\omega_0\tau^{(j)}}$ 是 $\boldsymbol{A}(0)$ 关于 $\mathrm{i}\tau^{(j)}\omega_0$ 的特征向量，即

$$\boldsymbol{A}(0)\boldsymbol{q}(\theta)=\mathrm{i}\tau^{(j)}\omega_0\boldsymbol{q}(\theta)$$

计算得

$$\alpha=-\frac{By^*}{(B+x^*)x^*}+\frac{(\mathrm{i}\omega_0+RG)(B+x^*)\beta}{Rx^*}$$

$$\beta=\frac{R(B+x^*)(x^*\mathrm{e}^{-\mathrm{i}\omega_0\tau}-\mathrm{i}\omega_0)+Ry^*}{(B+x^*)(\mathrm{i}\omega_0+RG+x^*HR)}$$

同理，设 $\boldsymbol{q}^*(s)=D(1,\alpha^*,\beta^*)\mathrm{e}^{\mathrm{i}\theta\omega_0\tau^{(j)}}$ 是 $\boldsymbol{A}^*$ 关于 $-\mathrm{i}\tau^{(j)}\omega_0$ 的特征向量，计算得

$$\alpha^*=\frac{(D+y^*)[x^*H+(\mathrm{i}\omega_0+RG)]\beta^*}{Qy^*}$$

$$\beta^*=\frac{(B+x^*)^2(-\mathrm{i}\omega_0-x^*\mathrm{e}^{\mathrm{i}\omega_0\tau})-x^*y^*-x^*EBH(D+y^*)}{EB(D+y^*)(\mathrm{i}\omega_0+RG)+RBy^*}$$

由双线性内积(8)，可将 $\boldsymbol{q}^*$ 规范化. 令 $\boldsymbol{x}_t$ 为(5)式在 $\mu=0$ 时的解，定义

$$z(t)=\langle\boldsymbol{q}^*,\boldsymbol{x}_t\rangle\qquad \boldsymbol{W}(t,\theta)=\boldsymbol{x}_t(\theta)-2\mathrm{Re}\{z(t)\boldsymbol{q}(\theta)\} \tag{9}$$

在中心流形 C_0 上，有

$$\boldsymbol{W}(t,\theta)=\boldsymbol{W}(z(t),\overline{z(t)},\theta)$$

其中

$$\boldsymbol{W}(z,\bar{z},\theta)=\boldsymbol{W}_{20}(\theta)\frac{z^2}{2}+\boldsymbol{W}_{11}(\theta)z\bar{z}+\boldsymbol{W}_{02}(\theta)\frac{\bar{z}^2}{2}+\boldsymbol{W}_{30}(\theta)\frac{z^3}{6}+\boldsymbol{L}$$

$$\boldsymbol{W}_{20}=(W_{20}^{(1)},W_{20}^{(2)},W_{20}^{(3)})\qquad \boldsymbol{W}_{11}=(W_{11}^{(1)},W_{11}^{(2)},W_{11}^{(3)})\qquad \boldsymbol{W}_{02}=(W_{02}^{(1)},W_{02}^{(2)},W_{02}^{(3)})$$

z 和 $\bar{z}$ 是中心流形 C_0 在 $\boldsymbol{q}^*$ 和 $\overline{\boldsymbol{q}^*}$ 方向上的局部坐标. 方程(5)的解为 $x_t\in C_0$. 因为 $\mu=0$，有

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\mathrm{i}\omega_0\tau^{(j)}z+\langle\boldsymbol{q}^*(\theta),\boldsymbol{F}(0,\boldsymbol{W}(z,\bar{z},\theta)+2\mathrm{Re}\{z\boldsymbol{q}(\theta)\})\rangle=$$

$$\mathrm{i}\omega_0\tau^{(j)}z+\langle\overline{\boldsymbol{q}^*}(0)\boldsymbol{F}_0(z,\bar{z})\rangle=\mathrm{i}\omega_0\tau^{(j)}z+g(z,\bar{z}) \tag{10}$$

其中

$$g(z,\bar{z})=\overline{\boldsymbol{q}^*}(0)\boldsymbol{F}_0(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots \tag{11}$$

由(9)式得

$$\boldsymbol{x}_t(\theta)=(x_{1t}(\theta),x_{2t}(\theta),x_{3t}(\theta))^{\mathrm{T}}=\boldsymbol{W}(t,\theta)+z\boldsymbol{q}(\theta)+\bar{z}\bar{\boldsymbol{q}}(\theta)$$

故

$$x_{1t}(0)=z+\bar{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\overline{z^2}}{2}+o(|(z,\bar{z})|^3)$$

$$x_{1t}(-1)=z\mathrm{e}^{-\mathrm{i}\omega_0\tau^{(j)}}+\bar{z}\mathrm{e}^{\mathrm{i}\omega_0\tau^{(j)}}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\frac{\overline{z^2}}{2}+o(|(z,\bar{z})|^3)$$

$$x_{2t}(0)=z\alpha+\bar{z}\bar{\alpha}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+o(|(z,\bar{z})|^3)$$

$$x_{3t}(0)=z\beta+\bar{z}\bar{\beta}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\bar{z}+W_{02}^{(3)}(0)\frac{\bar{z}^2}{2}+o(|(z,\bar{z})|^3)$$

根据(10)式,得

$$g(z,\bar{z})=\overline{\langle \boldsymbol{q}^*(0),\boldsymbol{F}_0(z,\bar{z})\rangle}=\bar{D}\tau^{(j)}(1,\overline{\alpha^*},\overline{\beta^*})(m_1,m_2,m_3)^{\mathrm{T}}$$

其中

$$m_1=\frac{B\phi_1(0)}{(B+x^*)^2}\left(1-\frac{\phi_1(0)}{B+x^*}\right)\left(\frac{y^*\phi_1(0)}{B+x^*}-\phi_2(0)\right)-\phi_1(0)\phi_1(-1)-H\phi_1(0)\phi_3(0)$$

$$m_2=\frac{QEB\phi_1(0)}{(B+x^*)^2}\left(\frac{\phi_1(0)}{B+x^*}-1\right)\left(\frac{y^*\phi_1(0)}{B+x^*}-\phi_2(0)\right)+\frac{QD\phi_2(0)}{(D+y^*)^2}\left(1-\frac{\phi_2(0)}{D+y^*}\right)\left(\frac{z^*\phi_2(0)}{D+y^*}-\phi_3(0)\right)$$

$$m_3=\frac{RBy^*\phi_1^2(0)}{(B+x^*)^3}\left(\frac{\phi_1(0)}{B+x^*}-1\right)+\frac{RB\phi_1(0)\phi_2(0)}{(B+x^*)^2}\left(1-\frac{\phi_1(0)}{B+x^*}\right)$$

将 $x_{1t}(0)$,$x_{1t}(-1)$,$x_{2t}(0)$,$x_{3t}(0)$ 带入 $g(z,\bar{z})$,与(11)式比较系数,得

$$\begin{aligned}g_{20}=&2\bar{D}\tau^{(j)}\left[\frac{By^*-B\alpha(B+x^*)}{(B+x^*)^3}(1-R\overline{\beta^*})-\mathrm{e}^{-\mathrm{i}\omega_0\tau}+H\beta\right]+\\&\overline{\alpha^*}\left(\frac{-EQBy^*}{(B+x^*)^3}+\frac{QDz^*\alpha^2}{(D+y^*)^3}-\frac{QD\alpha\beta}{(D+y^*)^2}+\frac{QEB\alpha}{(B+x^*)^2}\right)\end{aligned}$$

$$\begin{aligned}g_{11}=&2\bar{D}\tau^{(j)}\left[\frac{By^*-B\mathrm{Re}\{\alpha\}(B+x^*)}{(B+x^*)^3}(1-R\overline{\beta^*})-\frac{\mathrm{e}^{-\mathrm{i}\omega_0\tau}+\mathrm{e}^{\mathrm{i}\omega_0\tau}}{2}-\frac{H(\beta+\bar{\beta})}{2}\right]+\\&\overline{\alpha^*}\left(\frac{-EQBy^*}{(B+x^*)^3}+\frac{QDz^*\alpha\bar{\alpha}}{(D+y^*)^3}-\frac{QD\mathrm{Re}\{\bar{\alpha}\beta\}}{(D+y^*)^2}+\frac{QEB\mathrm{Re}\{\alpha\}}{(B+x^*)^2}\right)\end{aligned}$$

$$\begin{aligned}g_{02}=&2\bar{D}\tau^{(j)}\left[\frac{By^*-B\alpha(B+x^*)}{(B+x^*)^3}(1-R\overline{\beta^*})-\mathrm{e}^{\mathrm{i}\omega_0\tau}-H\bar{\beta}\right]+\\&\overline{\alpha^*}\left(\frac{-EQBy^*}{(B+x^*)^3}+\frac{QDz^*\overline{\alpha^2}}{(D+y^*)^3}-\frac{QD\bar{\alpha}\bar{\beta}}{(D+y^*)^2}+\frac{QEB\bar{\alpha}}{(B+x^*)^2}\right)\end{aligned}$$

$$\begin{aligned}g_{21}=&W_{11}^{(1)}(0)\left[\frac{2By^*-B\alpha(B+x^*)}{(B+x^*)^3}-\mathrm{e}^{-\mathrm{i}\omega_0\tau}-H\beta+\frac{\overline{\alpha^*}(EQB\alpha(B+x^*)-2EQBy^*)}{(B+x^*)^3}\right]+\\&W_{20}^{(1)}(0)\left[\frac{2By^*-B\bar{\alpha}(B+x^*)}{2(B+x^*)^3}+\frac{\mathrm{e}^{\mathrm{i}\omega_0\tau}+H\beta}{2}+\frac{\overline{\alpha^*}(EQB\bar{\alpha}(B+x^*))-2EQBy^*}{2(B+x^*)^3}\right]+\\&W_{11}^{(2)}(0)\left[\frac{-B}{2(B+x^*)^2}+\frac{QD\overline{\alpha^*}(2z^*\alpha-\beta)}{(D+y^*)^3}+\frac{\overline{\alpha^*}EBQ+RB\overline{\beta^*}}{(B+x^*)^2}\right]+\\&W_{20}^{(2)}(0)\left[\frac{-B}{2(B+x^*)^2}+\frac{QD\overline{\alpha^*}(2z^*\alpha-\beta)}{2(D+y^*)^3}+\frac{\overline{\alpha^*}EBQ+RB\overline{\beta^*}}{2(B+x^*)^2}\right]-\\&W_{11}^{(3)}(0)\left[H+\frac{\overline{\alpha^*}DQ\alpha}{(D+y^*)^3}\right]-W_{20}^{(3)}(0)\left[\frac{H}{2}+\frac{\overline{\alpha^*}DQ\bar{\alpha}}{2(D+y^*)^3}\right]-W_{11}^{(1)}(0)-\frac{1}{2}W_{20}^{(1)}(-1)+\\&\frac{\bar{\alpha}QEB[3y^*-(2\alpha+\bar{\alpha})(B+x^*)]}{(B+x^*)^4}-\frac{\overline{\beta^*}RB(\bar{\alpha}+2\alpha)}{(B+x^*)^3}\end{aligned}$$

为了确定 g_{21},需要计算 $\boldsymbol{W}_{20}(\theta)$,$\boldsymbol{W}_{11}(\theta)$. 由方程(9),(10)得

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t}=&\frac{\mathrm{d}\boldsymbol{x}_t}{\mathrm{d}t}-\frac{\mathrm{d}z}{\mathrm{d}t}\boldsymbol{q}-\frac{\mathrm{d}\bar{z}}{\mathrm{d}t}\bar{\boldsymbol{q}}=\begin{cases}\boldsymbol{AW}-2\mathrm{Re}\{\overline{\boldsymbol{q}^*}(0)\boldsymbol{F}_0\boldsymbol{q}(\theta)\} & \theta\in[-1,0)\\ \boldsymbol{AW}-2\mathrm{Re}\{\overline{\boldsymbol{q}^*}(0)\boldsymbol{F}_0\boldsymbol{q}(0)]+\boldsymbol{F}_0 & \theta=0\end{cases}\\=&\boldsymbol{AW}+\boldsymbol{H}(z,\bar{z},\theta)\end{aligned}\tag{12}$$

其中

$$\boldsymbol{H}(z,\bar{z},\theta)=\boldsymbol{H}_{20}(\theta)\frac{z^2}{2}+\boldsymbol{H}_{11}(\theta)z\bar{z}+\boldsymbol{H}_{02}\frac{\bar{z}^2}{2}+\cdots \tag{13}$$

注意到，在接近原点的中心流形 C_0 上，$\boldsymbol{W}=\boldsymbol{W}_z\dfrac{\mathrm{d}z}{\mathrm{d}t}+\boldsymbol{W}_{\bar{z}}\dfrac{\mathrm{d}\bar{z}}{\mathrm{d}t}$. 由(12),(13)式得

$$(\boldsymbol{A}-2\mathrm{i}\tau^{(j)}\omega_0)\boldsymbol{W}_{20}=-\boldsymbol{H}_{20}(\theta)\qquad \boldsymbol{A}\boldsymbol{W}_{11}(\theta)=-\boldsymbol{H}_{11}(\theta) \tag{14}$$

对于任意的 $\theta\in[-1,0)$，

$$\boldsymbol{H}(z,\bar{z},\theta)=-\bar{\boldsymbol{q}}^*(0)\boldsymbol{F}_0\boldsymbol{q}(\theta)-\boldsymbol{q}^*(0)\bar{\boldsymbol{F}}_0\bar{\boldsymbol{q}}(\theta)=-g\boldsymbol{q}(\theta)-\bar{g}\bar{\boldsymbol{q}}(\theta) \tag{15}$$

与(13)式比较系数，得

$$\boldsymbol{H}_{20}(\theta)=-g_{20}\boldsymbol{q}(\theta)-\bar{g}_{02}\bar{\boldsymbol{q}}(\theta)\qquad \boldsymbol{H}_{11}(\theta)=-g_{11}\boldsymbol{q}(\theta)-\bar{g}_{11}\bar{\boldsymbol{q}}(\theta)$$

由(12),(14)式及矩阵 $\boldsymbol{A}$，得到

$$\frac{\mathrm{d}\boldsymbol{W}_{20}}{\mathrm{d}t}=2\mathrm{i}\omega_0\tau^{(j)}\boldsymbol{W}_{20}(\theta)+g_{20}\boldsymbol{q}(\theta)+\bar{g}_{02}\bar{\boldsymbol{q}}(\theta) \tag{16}$$

因 $\boldsymbol{q}(\theta)=(1,\alpha,\beta)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\theta\omega_0\tau^{(j)}}$，有

$$\boldsymbol{W}_{20}(\theta)=\frac{\mathrm{i}g_{20}}{\tau^{(j)}\omega_0}\boldsymbol{q}(0)\mathrm{e}^{\mathrm{i}\theta\omega_0\tau^{(j)}}+\frac{\mathrm{i}\bar{g}_{11}}{3\tau^{(j)}\omega_0}\bar{\boldsymbol{q}}(0)\mathrm{e}^{-\mathrm{i}\theta\omega_0\tau^{(j)}}+\boldsymbol{E}_1\mathrm{e}^{2\mathrm{i}\theta\omega_0\tau^{(j)}} \tag{17}$$

其中 $\boldsymbol{E}_1=(E_1^{(1)},E_1^{(2)},E_1^{(3)})^{\mathrm{T}}\in\mathbb{R}^3$ 为常向量，同理可得

$$\boldsymbol{W}_{11}(\theta)=\frac{\mathrm{i}g_{11}}{-\tau^{(j)}\omega_0}\boldsymbol{q}(0)\mathrm{e}^{\mathrm{i}\theta\omega_0\tau^{(j)}}+\frac{\mathrm{i}\bar{g}_{11}}{\tau^{(j)}\omega_0}\bar{\boldsymbol{q}}(0)\mathrm{e}^{-\mathrm{i}\theta\omega_0\tau^{(j)}}+\boldsymbol{E}_2 \tag{18}$$

其中 $\boldsymbol{E}_2=(E_2^{(1)},E_2^{(2)},E_2^{(3)})^{\mathrm{T}}\in\mathbb{R}^3$ 为常向量. 下面计算 $\boldsymbol{E}_1,\boldsymbol{E}_2$. 由(14)式及 $\boldsymbol{A}$ 的定义，得

$$\int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{W}_{20}=2\mathrm{i}\tau^{(j)}\omega_0\boldsymbol{W}_{20}(0)-\boldsymbol{H}_{20}(0)\qquad \int_{-1}^{0}\mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{W}_{11}=-\boldsymbol{H}_{11}(0)$$

其中 $\boldsymbol{\eta}(\theta)=\boldsymbol{\eta}(\theta,0)$，则

$$\boldsymbol{H}_{20}(\theta)=-g_{20}\boldsymbol{q}(\theta)-\bar{g}_{02}\bar{\boldsymbol{q}}(\theta)+2\tau^{(j)}\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$$

$$\boldsymbol{H}_{11}(\theta)=-g_{11}\boldsymbol{q}(\theta)-\bar{g}_{11}\bar{\boldsymbol{q}}(\theta)+2\tau^{(j)}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

其中

$$d_1=\frac{By^*-B\alpha(B+x^*)}{(B+x^*)^3}-\mathrm{e}^{-\mathrm{i}\omega_0\tau}-H\beta$$

$$d_2=-\frac{EQBy^*}{(B+x^*)^3}+\frac{QDz^*\alpha^2}{(D+y^*)^3}-\frac{QD\alpha\beta}{(D+y^*)^2}+\frac{QEB\alpha}{(B+x^*)^2}$$

$$d_3=\frac{RB\alpha}{(B+x^*)^2}-\frac{RBy^*}{(B+x^*)^3}$$

$$c_1=\frac{By^*-B\mathrm{Re}\{\alpha\}(B+x^*)}{(B+x^*)^3}-\frac{\mathrm{e}^{\mathrm{i}\omega_0\tau}+\mathrm{e}^{-\mathrm{i}\omega_0\tau}}{2}-H\mathrm{Re}\{\beta\}$$

$$c_2=\frac{-EQBy^*}{(B+x^*)^3}+\frac{QDz^*\bar{\alpha}\alpha}{(D+y^*)^3}-\frac{QD\mathrm{Re}[\bar{\alpha}\beta]}{(D+y^*)^2}+\frac{EQB\mathrm{Re}[\alpha]}{(B+x^*)^2}$$

$$c_3=\frac{RB\mathrm{Re}\{\alpha\}}{(B+x^*)^2}-\frac{RBy^*}{(B+x^*)^3}$$

注意到

$$\left(\mathrm{i}\tau^{(j)}\omega_0\boldsymbol{I}-\int_{-1}^{0}\mathrm{e}^{\mathrm{i}\theta\omega_0\tau^{(j)}}\mathrm{d}\boldsymbol{\eta}(\theta)\right)\boldsymbol{q}(0)=0\qquad \left(-\mathrm{i}\tau^{(j)}\omega_0\boldsymbol{I}-\int_{-1}^{0}\mathrm{e}^{-\mathrm{i}\theta\omega_0\tau^{(j)}}\mathrm{d}\boldsymbol{\eta}(\theta)\right)\bar{\boldsymbol{q}}(0)=0$$

得方程

$$\begin{pmatrix} 2\mathrm{i}\omega_0 - \frac{x^* y^*}{(B+x^*)^2} - x^* \mathrm{e}^{-2\mathrm{i}\omega_0} & \frac{x^*}{B+x^*} & x^* H \\ -\frac{EQBy^*}{(B+x^*)^2} & 2\mathrm{i}\omega_0 - \frac{Qy^* z^*}{(D+y^*)^2} & -\frac{Qy^*}{D+y^*} \\ -\frac{RBy^*}{(B+x^*)^2} & -\frac{Rx^*}{B+x^*} & 2\mathrm{i}\omega_0 + RG \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x^* y^*}{(B+x^*)^2} - x^* & \frac{x^*}{B+x^*} & x^* H \\ -\frac{EQBy^*}{(B+x^*)^2} & -\frac{Qy^* z^*}{(D+y^*)^2} & -\frac{Qy^*}{D+y^*} \\ -\frac{RBy^*}{(B+x^*)^2} & -\frac{Rx^*}{B+x^*} & RG \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = 2 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

解方程可得 $E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_2^{(1)}, E_2^{(2)}, E_2^{(3)}$.

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The Hopf Bifurcation's Direction and Stability of the Dynastic Cycle Model

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Abstract: The positive equilibrium's stability changes when the delay arrives at or crosses a sequence of critical values in the dynamical model of the dynastic cycle with delay. Employing the normal form theory and the center manifold argument in this paper, we give the explicit formulas that determine the direction and other properties of bifurcation periodic solutions.

Key words: Hopf bifurcation; stability; periodic solution

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