第 38 卷第 2 期　　西南大学学报(自然科学版)　　2016 年 2 月
Vol. 38 No. 2　　Journal of Southwest University (Natural Science Edition)　　Feb. 2016

DOI: 10.13718/j.cnki.xdzk.2016.02.011

弱(ψ, ϕ)压缩映象在偏矩度量空间中的公共不动点定理[①]

姚　婷[1]，　邓　磊[1]，　杨明歌[2]

1. 西南大学 数学与统计学院，重庆 400715；2. 洛阳师范学院 数学科学学院，河南 洛阳 471022

摘要：将改进的带有 4 个自映象的弱(ψ, ϕ)压缩映象引入到偏矩度量空间中，证明了此映象在该空间中的公共不动点定理. 该结果改进和推广了近期的相关结果.

关　键　词：弱(ψ, ϕ)压缩映象；偏矩度量空间；公共不动点

中图分类号：O177.91　　**文献标志码**：A　　**文章编号**：1673-9868(2016)02-0064-08

度量空间在研究过程中已被推广为许多其它形式的度量空间. 文献[1]给出了度量空间的定义；文献[2]用四角不等式代替三角不等式，定义了广义(矩)度量空间；文献[3]定义了偏度量空间；文献[4]结合偏度量和矩度量的定义，给出了偏矩度量空间和该空间中的序列收敛以及空间完备的定义.

文献[5]定义了(ϕ)-压缩映象；文献[6]定义了弱(ϕ)-压缩映象；文献[7]在模糊度量空间中定义了广义压缩映象. 这类压缩映象在许多空间中被研究并得到推广，文献[8]将这类压缩映象推广为弱(ψ, ϕ)压缩映象.

本文首先证明了文献[4]中介绍的偏矩度量空间的一个特殊性质，其次将弱(ψ, ϕ)压缩映象改进，证明了该映象在此空间中公共不动点的存在及唯一性定理.

1　预备知识

定义 1[4]　设 X 是一非空集合，映射 ρ: $X \times X \longrightarrow \mathbb{R}$. 如果对于任意的 $x, y \in X$，不同的 $z, w \in X\backslash\{x, y\}$，满足：

(i) $\rho(x, y) \geqslant 0$；

(ii) $\rho(x, y)=\rho(x, x)=\rho(y, y)$ 当且仅当 $x=y$；

(iii) $\rho(x, x) \leqslant \rho(x, y)$；

(iv) $\rho(x, y)=\rho(y, x)$；

(v) $\rho(x, y) \leqslant \rho(x, z)+\rho(z, w)+\rho(w, y)-\rho(z, z)-\rho(w, w)$.

则称 ρ 是 X 的偏矩度量，称(X, ρ)是偏矩度量空间.

定义 2[4]　设(X, ρ)是偏矩度量空间，$\{x_n\}$ 是 X 中的一个序列，那么：

(i) 如果存在 $x \in X$，使得 $\lim\limits_{n\to\infty} \rho(x_n, x)=\rho(x, x)$，则称 $\{x_n\}$ 收敛于 x；

(ii) 如果 $\lim\limits_{n,m\to\infty} \rho(x_n, x_m)$ 存在且有限，则称 $\{x_n\}$ 是柯西列；

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① 收稿日期：2015-05-09

基金项目：国家自然科学基金项目(11226228)；河南省高等学校重点科研项目(15A110036).

作者简介：姚　婷(1990-)，女，河南南阳人，硕士研究生，主要从事非线性泛函分析的研究.

通信作者：邓　磊，教授.

(iii) 如果对 X 中任意柯西列$\{x_n\}$，都存在 $x\in X$，使得

$$\lim_{n,m\to\infty}\rho(x_n, x_m)=\lim_{n\to\infty}\rho(x_n, x)=\rho(x, x)$$

则称偏矩度量空间(X, ρ)是完备的.

定义 3[8] 设 Ψ 表示所有映射 $\psi:[0, \infty)\longrightarrow[0, \infty)$，且满足：

(i) ψ 是连续的；

(ii) $\psi(t)=0$ 当且仅当 $t=0$.

引理 1[8] 设 X 是非空集合，f, g, R, S 是 X 到 X 的 4 个自映象. 若映象对(f, S)和(g, R)有唯一的重合点，且分别弱相容，那么 f, g, R, S 有唯一公共不动点(重合点即是公共不动点).

证 设 $z\in X$ 是映象对(f, S)和(g, R)的重合点，那么存在 $w\in X$，使得 $w=fz=Sz$. 因为(f, S)弱相容，所以 $Sfz=fSz$，即有 $Sw=fw$，从而 $z, w\in X$ 均为重合点. 又因为重合点是唯一的，所以 $w=z$，也即 $z=fz=Sz$. 同理可证 $z=gz=Rz$. 因此，$z\in X$ 是 f, g, R, S 的唯一公共不动点.

引理 2 设(X, ρ)是偏矩度量空间，$\{y_k\}$是 X 中的序列，若有

$$\lim_{k\to\infty}\rho(y_k, y_{k+1})=0$$

那么对于任意的 $n, m, p, q\in\mathbb{N}_+$，且 $n\neq m$，$p\neq q$，有

$$\lim_{n,m\to\infty}\rho(y_{2n}, y_{2m})=\lim_{p,q\to\infty}\rho(y_{2p+1}, y_{2q+1})=\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})$$

证 因为偏矩度量空间中 $\rho(x, x)\leqslant\rho(x, y)$，所以

$$\lim_{k\to\infty}\rho(y_k, y_k)\leqslant\lim_{k\to\infty}\rho(y_k, y_{k+1})=0$$

首先，对于 $\rho(y_{2n}, y_{2n+2})$ 和 $\rho(y_{2n-1}, y_{2n+1})$，由定义 1(v)，有：

$$\rho(y_{2n}, y_{2n+2})\leqslant\rho(y_{2n}, y_{2n-1})+\rho(y_{2n-1}, y_{2n+1})+\rho(y_{2n+1}, y_{2n+2})-\rho(y_{2n-1}, y_{2n-1})-\rho(y_{2n+1}, y_{2n+1})$$

$$\rho(y_{2n-1}, y_{2n+1})\leqslant\rho(y_{2n-1}, y_{2n})+\rho(y_{2n}, y_{2n+2})+\rho(y_{2n+1}, y_{2n+2})-\rho(y_{2n}, y_{2n})-\rho(y_{2n+2}, y_{2n+2})$$

令 $n\to\infty$，可得：

$$\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})\leqslant\lim_{n\to\infty}\rho(y_{2n-1}, y_{2n+1})$$

$$\lim_{n\to\infty}\rho(y_{2n-1}, y_{2n+1})\leqslant\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})$$

所以

$$\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})=\lim_{n\to\infty}\rho(y_{2n-1}, y_{2n+1})$$

其次，对于 $\rho(y_{2n}, y_{2n+2})$ 和 $\rho(y_{2n-1}, y_{2n+3})$，有：

$$\rho(y_{2n}, y_{2n+2})\leqslant\rho(y_{2n}, y_{2n-1})+\rho(y_{2n-1}, y_{2n+3})+\rho(y_{2n+3}, y_{2n+2})-\rho(y_{2n-1}, y_{2n-1})-\rho(y_{2n+3}, y_{2n+3})$$

$$\rho(y_{2n-1}, y_{2n+3})\leqslant\rho(y_{2n-1}, y_{2n})+\rho(y_{2n}, y_{2n+2})+\rho(y_{2n+2}, y_{2n+3})-\rho(y_{2n}, y_{2n})-\rho(y_{2n+2}, y_{2n+2})$$

令 $n\to\infty$，可得：

$$\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})\leqslant\lim_{n\to\infty}\rho(y_{2n-1}, y_{2n+3})$$

$$\lim_{n\to\infty}\rho(y_{2n-1}, y_{2n+3})\leqslant\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})$$

因此，

$$\lim_{n\to\infty}\rho(y_{2n}, y_{2n+2})=\lim_{n\to\infty}\rho(y_{2n-1}, y_{2n+3})$$

再次，对于 $\rho(y_{2n}, y_{2n+2})$ 和 $\rho(y_{2n-2}, y_{2n+4})$，两次利用定义 1，有：

$$\rho(y_{2n}, y_{2n+2})\leqslant\rho(y_{2n}, y_{2n-1})+\rho(y_{2n-1}, y_{2n+3})+\rho(y_{2n+3}, y_{2n+2})-\rho(y_{2n-1}, y_{2n-1})-\rho(y_{2n+3}, y_{2n+3})\leqslant$$
$$\rho(y_{2n}, y_{2n-1})+\rho(y_{2n+2}, y_{2n+3})+\rho(y_{2n-2}, y_{2n-1})+\rho(y_{2n-2}, y_{2n+4})+\rho(y_{2n+4}, y_{2n+3})-\rho(y_{2n-1}, y_{2n-1})-$$

$$\rho(y_{2n+1}, y_{2n+1}) - \rho(y_{2n-2}, y_{2n-2}) - \rho(y_{2n+4}, y_{2n+4})$$
$$\rho(y_{2n-2}, y_{2n+4}) \leqslant \rho(y_{2n-2}, y_{2n-1}) + \rho(y_{2n-1}, y_{2n+3}) + \rho(y_{2n+3}, y_{2n+4}) - \rho(y_{2n-1}, y_{2n-1}) - \rho(y_{2n+3}, y_{2n+3}) \leqslant \rho(y_{2n}, y_{2n-1}) + \rho(y_{2n+3}, y_{2n+4}) + \rho(y_{2n-1}, y_{2n}) + \rho(y_{2n}, y_{2n+2}) + \rho(y_{2n+2}, y_{2n+3}) - \rho(y_{2n-1}, y_{2n-1}) - \rho(y_{2n+3}, y_{2n+3}) - \rho(y_{2n}, y_{2n}) - \rho(y_{2n+2}, y_{2n+2})$$

令 $n \to \infty$，可得：

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n+2}) \leqslant \lim_{n\to\infty} \rho(y_{2n-2}, y_{2n+4})$$
$$\lim_{n\to\infty} \rho(y_{2n-2}, y_{2n+4}) \leqslant \lim_{n\to\infty} \rho(y_{2n}, y_{2n+2})$$

因此，

$$\lim_{n\to\infty} \rho(y_{2n-2}, y_{2n+4}) = \lim_{n\to\infty} \rho(y_{2n}, y_{2n+2})$$

依此类推，对于下标差为偶数的 y_p, y_q，有限次利用定义 1(Ⅴ)，同样可以得到

$$\lim_{p,q\to\infty} \rho(y_p, y_q) = \lim_{n\to\infty} \rho(y_{2n}, y_{2n+2})$$

因此，对于任意的 $n, m, p, q \in \mathbb{N}_+$，且 $n \neq m$，$p \neq q$，有

$$\lim_{n,m\to\infty} \rho(y_{2n}, y_{2m}) = \lim_{p,q\to\infty} \rho(y_{2p+1}, y_{2q+1}) = \lim_{n\to\infty} \rho(y_{2n}, y_{2n+2})$$

2　主要结果

定理 1　设(X, ρ)是偏矩度量空间，4 个自映象 f, g, R, S：$X \longrightarrow X$ 满足：$f(X) \subseteq R(X)$，$g(X) \subseteq S(X)$，其中 $R(X)$ 和 $S(X)$ 是 X 中的完备子集. 对于 $\psi, \phi \in \Psi$，且 ψ 是非减的，有

$$\psi(\rho(fx, gy)) \leqslant \psi(M(x, y)) - \phi(M(x, y)) \tag{1}$$

成立，其中

$$M(x, y) = \max\left\{\rho(Sx, Ry), \rho(Sx, fx), \rho(Ry, gy), \frac{\rho(Rg^{-1}Sx, gy) + \rho(Ry, fx)}{3}\right\}$$

若映象对(f, S)和(g, R)分别弱相容，且 f, g 是相对弱增的，则映象对(f, S)和(g, R)有唯一重合点，且 f, g, R, S 有唯一公共不动点 z，满足 $\rho(z, z) = 0$.

证　对于任意 $x_0 \in X$，由 $f(X) \subseteq R(X)$，$g(X) \subseteq S(X)$，定义序列$\{x_n\}$，$\{y_n\}$ 如下：

$$\begin{cases} y_{2n+1} = Rx_{2n+1} = fx_{2n} \\ y_{2n+2} = Sx_{2n+2} = gx_{2n+1} \end{cases} \tag{2}$$

因为 f, g 是相对弱增的，假设 $Rx_1 = fx_0 \leqslant gfx_0 = gx_1 = Sx_2$. 因此，对于任意的 $n \geqslant 0$，$Rx_{2n+1} \leqslant Sx_{2n+2}$.

步骤 1　证明$\lim\limits_{k\to\infty} \rho(y_k, y_{k+1}) = 0$.

特别地，证明当 $k = 2n$ 时，有

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n+1}) = 0$$

或者

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n-1}) = 0$$

当 $k = 2n$ 时，由(1)式，有

$$\psi(\rho(y_{2n+1}, y_{2n+2})) = \psi(\rho(fx_{2n}, gx_{2n+1})) \leqslant \psi(M(x_{2n}, x_{2n+1})) - \phi(M(x_{2n}, x_{2n+1})) \tag{3}$$

其中

$$M(x_{2n}, x_{2n+1}) = \max\left\{\rho(Sx_{2n}, Rx_{2n+1}), \rho(Sx_{2n}, fx_{2n}), \rho(Rx_{2n+1}, gx_{2n+1}), \frac{\rho(Rg^{-1}Sx_{2n}, gx_{2n+1}) + \rho(Rx_{2n+1}, fx_{2n})}{3}\right\} = \max\left\{\rho(y_{2n}, y_{2n+1}), \rho(y_{2n}, y_{2n+1}), \rho(y_{2n+1}, y_{2n+2}), \frac{\rho(y_{2n-1}, y_{2n+2}) + \rho(y_{2n+1}, y_{2n+1})}{3}\right\} =$$

$$\max\left\{\rho(y_{2n}, y_{2n+1}), \rho(y_{2n+1}, y_{2n+2}), \frac{\rho(y_{2n-1}, y_{2n+2})+\rho(y_{2n+1}, y_{2n+1})}{3}\right\} \tag{4}$$

若序列$\{y_n\}$中存在相邻两项相等，特别地，

$$y_{2n}=y_{2n+1}$$

此时有

$$\rho(y_{2n}, y_{2n+1})=\rho(y_{2n+1}, y_{2n+1})\leqslant\rho(y_{2n+1}, y_{2n+2})$$

从而由(4)式得

$$M(x_{2n}, x_{2n+1})=\max\left\{\rho(y_{2n+1}, y_{2n+2}), \frac{\rho(y_{2n-1}, y_{2n+2})+\rho(y_{2n+1}, y_{2n+1})}{3}\right\} \tag{5}$$

如果

$$M(x_{2n}, x_{2n+1})=\rho(y_{2n+1}, y_{2n+2})$$

带入(3)式，可得

$$\psi(\rho(y_{2n+1}, y_{2n+2}))\leqslant\psi(\rho(y_{2n+1}, y_{2n+2}))-\phi(\rho(y_{2n+1}, y_{2n+2}))$$

因此，

$$\phi(\rho(y_{2n+1}, y_{2n+2}))=0$$

又因为$\phi\in\Psi$，所以

$$\rho(y_{2n+1}, y_{2n+2})=0$$

由定义1，可得

$$y_{2n}=y_{2n+1}=y_{2n+2}$$

因此，序列$\{y_n\}$从相等项开始每一项均与前一项相等，且

$$\rho(y_{2n+1}, y_{2n+2})=0$$

所以

$$\lim_{n\to\infty}\rho(y_{2n}, y_{2n+1})=0$$

如果(5)式中，

$$M(x_{2n}, x_{2n+1})=\frac{\rho(y_{2n-1}, y_{2n+2})+\rho(y_{2n+1}, y_{2n+1})}{3}$$

由定义1，可得

$$\begin{aligned}M(x_{2n}, x_{2n+1})&=\frac{\rho(y_{2n-1}, y_{2n+2})+\rho(y_{2n+1}, y_{2n+1})}{3}\leqslant\\&\frac{\rho(y_{2n-1}, y_{2n})+\rho(y_{2n}, y_{2n+1})+\rho(y_{2n+1}, y_{2n+2})-\rho(y_{2n}, y_{2n})}{3}=\\&\frac{\rho(y_{2n-1}, y_{2n})+\rho(y_{2n+1}, y_{2n+2})}{3}\end{aligned} \tag{6}$$

将(6)式带入(3)式，可得

$$\begin{aligned}\psi(\rho(y_{2n+1}, y_{2n+2}))&\leqslant\psi\left(\frac{\rho(y_{2n-1}, y_{2n})+\rho(y_{2n+1}, y_{2n+2})}{3}\right)-\phi(M(x_{2n}, x_{2n+1}))\leqslant\\&\psi\left(\frac{\rho(y_{2n-1}, y_{2n})+\rho(y_{2n+1}, y_{2n+2})}{3}\right)\end{aligned} \tag{7}$$

因为ψ是非减的，所以

$$\rho(y_{2n+1}, y_{2n+2})\leqslant\frac{\rho(y_{2n-1}, y_{2n})+\rho(y_{2n+1}, y_{2n+2})}{3}$$

即

$$\rho(y_{2n+1}, y_{2n+2})\leqslant\frac{\rho(y_{2n-1}, y_{2n})}{2}$$

此时，对于$\rho(y_{2n+2}, y_{2n+3})$，由(1)式可得

$$\psi(\rho(y_{2n+2}, y_{2n+3})) = \psi(\rho(fx_{2n+2}, gx_{2n+1})) \leqslant \psi(M(x_{2n+2}, x_{2n+1})) - \phi(M(x_{2n+2}, x_{2n+1})) \tag{8}$$

其中

$$M(x_{2n+2}, x_{2n+1}) = \max\left\{\rho(Sx_{2n+2}, Rx_{2n+1}), \rho(Sx_{2n+2}, fx_{2n+2}), \rho(Rx_{2n+1}, gx_{2n+1}), \frac{\rho(Rg^{-1}Sx_{2n+2}, gx_{2n+1}) + \rho(Rx_{2n+1}, fx_{2n+2})}{3}\right\} = \max\left\{\rho(y_{2n+2}, y_{2n+1}), \rho(y_{2n+2}, y_{2n+3}), \frac{\rho(y_{2n}, y_{2n+3}) + \rho(y_{2n+2}, y_{2n+2})}{3}\right\} \tag{9}$$

由定义 1，以及 $y_{2n} = y_{2n+1}$，有

$$\rho(y_{2n}, y_{2n+3}) + \rho(y_{2n+2}, y_{2n+2}) \leqslant \rho(y_{2n}, y_{2n+1}) + \rho(y_{2n+1}, y_{2n+2}) + \rho(y_{2n+2}, y_{2n+3}) - \rho(y_{2n+1}, y_{2n+1}) = \rho(y_{2n+1}, y_{2n+2}) + \rho(y_{2n+2}, y_{2n+3})$$

所以

$$M(x_{2n+2}, x_{2n+1}) = \max\{\rho(y_{2n+1}, y_{2n+2}), \rho(y_{2n+2}, y_{2n+3})\} \tag{10}$$

如果

$$M(x_{2n+2}, x_{2n+1}) = \rho(y_{2n+2}, y_{2n+3})$$

带入(8)式，可得

$$\psi(\rho(y_{2n+2}, y_{2n+3})) \leqslant \psi(\rho(y_{2n+2}, y_{2n+3})) - \phi(\rho(y_{2n+2}, y_{2n+3})) \tag{11}$$

因此，

$$\phi(\rho(y_{2n+2}, y_{2n+3})) = 0$$

又因为 $\phi \in \Psi$，所以

$$\rho(y_{2n+2}, y_{2n+3}) = 0$$

再由定义 1，可得

$$y_{2n+2} = y_{2n+3}$$

此时

$$M(x_{2n+2}, x_{2n+1}) = \rho(y_{2n+2}, y_{2n+3}) = 0$$

也就有

$$\rho(y_{2n+2}, y_{2n+1}) = 0$$

所以

$$y_{2n+1} = y_{2n+2} = y_{2n+3}$$

所以，从相等项开始每一项均与前一项相等，且

$$\rho(y_{2n+1}, y_{2n+2}) = 0$$

所以

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n+1}) = 0$$

如果

$$M(x_{2n+2}, x_{2n+1}) = \rho(y_{2n+2}, y_{2n+1})$$

带入(8)式，可得

$$\psi(\rho(y_{2n+2}, y_{2n+3})) \leqslant \psi(\rho(y_{2n+1}, y_{2n+2})) - \phi(\rho(y_{2n+1}, y_{2n+2})) \leqslant \psi(\rho(y_{2n+1}, y_{2n+2})) \tag{12}$$

因为 ψ 是非减的，所以

$$\rho(y_{2n+2}, y_{2n+3}) < \rho(y_{2n+2}, y_{2n+1})$$

依此类推，可得$\{\rho(y_{2n+1}, y_{2n+2})\}$是递减的，又因为它有下界 0，所以$\{\rho(y_{2n+1}, y_{2n+2})\}$收敛，假设极限为 ε，即

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n+1}) = \varepsilon$$

将 $M(x_{2n+2}, x_{2n+1}) = \rho(y_{2n+2}, y_{2n+1})$ 式带入(8)式，并令 $n \to \infty$，可得 $\psi(\varepsilon) \leqslant \psi(\varepsilon) - \phi(\varepsilon)$，所以 $\phi(\varepsilon) = 0$. 由 $\phi \in \Psi$，可得 $\varepsilon = 0$，所以

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n+1}) = 0$$

若序列 $\{y_n\}$ 中任意相邻的两项不相等，特别地，

$$y_{2n-1} \neq y_{2n} \neq y_{2n+1}$$

如果(4)式中，

$$M(x_{2n}, x_{2n+1}) = \rho(y_{2n+1}, y_{2n+2})$$

带入(3)式，可得

$$\psi(\rho(y_{2n+1}, y_{2n+2})) \leqslant \psi(\rho(y_{2n+1}, y_{2n+2})) - \phi(\rho(y_{2n+1}, y_{2n+2})) \tag{13}$$

因此

$$\phi(\rho(y_{2n+1}, y_{2n+2})) = 0$$

从而

$$\rho(y_{2n+1}, y_{2n+2}) = 0$$

所以有

$$y_{2n} = y_{2n+1} = y_{2n+2}$$

与 $y_{2n} \neq y_{2n+1}$ 矛盾.

如果(4)式中，

$$M(x_{2n}, x_{2n+1}) = \rho(y_{2n}, y_{2n+1})$$

带入(3)式，可得

$$\psi(\rho(y_{2n+1}, y_{2n+2})) \leqslant \psi(\rho(y_{2n}, y_{2n+1})) - \phi(\rho(y_{2n}, y_{2n+1})) \leqslant \psi(\rho(y_{2n}, y_{2n+1})) \tag{14}$$

因为 ψ 是非减的，所以

$$\rho(y_{2n+1}, y_{2n+2}) \leqslant \rho(y_{2n}, y_{2n+1})$$

从而得到 $\{\rho(y_{2n}, y_{2n+1})\}$ 是递减的. 又因为它有下界 0，所以 $\{\rho(y_{2n}, y_{2n+1})\}$ 收敛. 同样的方法，可得

$$\lim_{n\to\infty} \rho(y_{2n}, y_{2n+1}) = 0$$

如果(4)式中，

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \frac{\rho(y_{2n-1}, y_{2n+2}) + \rho(y_{2n+1}, y_{2n+1})}{3} \leqslant \\ &\frac{\rho(y_{2n-1}, y_{2n}) + \rho(y_{2n}, y_{2n+1}) + \rho(y_{2n+1}, y_{2n+2}) - \rho(y_{2n}, y_{2n})}{3} \leqslant \\ &\frac{\rho(y_{2n-1}, y_{2n}) + \rho(y_{2n}, y_{2n+1}) + \rho(y_{2n+1}, y_{2n+2})}{3} \leqslant \\ &\frac{\rho(y_{2n-1}, y_{2n}) + 2\max\{\rho(y_{2n}, y_{2n+1}), \rho(y_{2n+1}, y_{2n+2})\}}{3} \end{aligned} \tag{15}$$

因为

$$\rho(y_{2n+1}, y_{2n+2}) < M(x_{2n}, x_{2n+1})$$

且

$$\rho(y_{2n}, y_{2n+1}) < M(x_{2n}, x_{2n+1})$$

所以

$$M(x_{2n}, x_{2n+1}) > \max\{\rho(y_{2n}, y_{2n+1}), \rho(y_{2n+1}, y_{2n+2})\}$$

因此有

$$\max\{\rho(y_{2n}, y_{2n+1}), \rho(y_{2n+1}, y_{2n+2})\} < \frac{\rho(y_{2n-1}, y_{2n}) + 2\max\{\rho(y_{2n}, y_{2n+1}), \rho(y_{2n+1}, y_{2n+2})\}}{3} \tag{16}$$

由(16)式，易知

$$\max\{\rho(y_{2n},y_{2n+1}),\rho(y_{2n+1},y_{2n+2})\}\leqslant\rho(y_{2n-1},y_{2n})$$

因此，必有 $\rho(y_{2n},y_{2n+1})\leqslant\rho(y_{2n-1},y_{2n})$. 同理可得，$\{\rho(y_{2n},y_{2n+1})\}$ 是递减的. 因此，同样有 $\lim\limits_{n\to\infty}\rho(y_{2n},y_{2n+1})=0$ 成立.

综上所述，当 $k=2n$ 时，有

$$\lim_{n\to\infty}\rho(y_{2n},y_{2n+1})=0$$

当 $k=2n+1$ 时，同理可得

$$\lim_{n\to\infty}\rho(y_{2n+1},y_{2n+2})=0$$

因此，$\lim\limits_{k\to\infty}\rho(y_k,y_{k+1})=0$.

步骤 2　证明$\{y_n\}$为柯西列.

由步骤 1 以及引理 2，可以得到

$$\lim_{n,m\to\infty}\rho(y_{2n},y_{2m})=\lim_{p,q\to\infty}\rho(y_{2p+1},y_{2q+1})=\lim_{n\to\infty}\rho(y_{2n},y_{2n+2})\tag{17}$$

由(17) 式，易知

$$\lim_{n,m\to\infty}\rho(y_{2n},y_{2n+2})-\rho(y_{2m},y_{2m+2})=0\tag{18}$$

所以，数列$\{\rho(y_{2n},y_{2n+2})\}$为柯西列. 又由于柯西数列极限存在，所以数列$\{\rho(y_{2n},y_{2n+2})\}$的极限存在，即存在常数 $r\geqslant0$，使得

$$\lim_{n\to\infty}\rho(y_{2n},y_{2n+2})=r$$

由引理 2，便有

$$\lim_{n,m\to\infty}\rho(y_{2n},y_{2m})=\lim_{p,q\to\infty}\rho(y_{2p+1},y_{2q+1})=\lim_{n\to\infty}\rho(y_{2n},y_{2n+2})=r$$

因此，由定义 2，得到$\{y_{2n}\}$和$\{y_{2n+1}\}$均为 X 中的柯西列.

因为 $R(X)$ 和 $S(X)$ 是 X 中的完备子集，所以$\{y_{2n}\}$和$\{y_{2n+1}\}$均收敛. 假设$\{y_{2n}\}$和$\{y_{2n+1}\}$收敛于不同的两点，其中$\{y_{2n}\}$收敛于 z^*，$\{y_{2n+1}\}$收敛于 z'，$z^*\neq z'$，且 $\rho(z^*,z^*)=\rho(z',z')=r$. 因此，

$$\lim_{n\to\infty}\rho(y_{2n},y_{2n+1})=\rho(z^*,z')=0$$

所以 $z^*=z'$，所以$\{y_n\}$为柯西列，且收敛于一点 z，且在该点处有 $\rho(z,z)=0$.

步骤 3　证明映象对(f,S)和(g,R)有重合点.

由$\{y_n\}$收敛于 z，以及

$$y_{2n+1}=Rx_{2n+1}=fx_{2n}\qquad y_{2n+2}=Sx_{2n+2}=gx_{2n+1}$$

所以存在 $u\in X$，使得 $Ru=z$，又由(1) 式，有

$$\psi(\rho(fx_{2n+2},gu))\leqslant\psi(M(x_{2n+2},u))-\phi(M(x_{2n+2},u))\tag{19}$$

其中

$$\begin{aligned}M(x_{2n+2},u)&=\max\left\{\rho(Sx_{2n+2},Ru),\rho(Sx_{2n+2},fx_{2n+2}),\rho(Ru,gu),\frac{\rho(RgSx_{2n+2},gu)+\rho(Ru,fx_{2n+2})}{3}\right\}=\\&\max\left\{\rho(y_{2n+2},z),\rho(y_{2n+2},y_{2n+3}),\rho(z,gu),\frac{\rho(y_{2n+1},gu)+\rho(z,y_{2n+3})}{3}\right\}\end{aligned}\tag{20}$$

将(20) 式带入(19) 式，并令 $n\to\infty$，有

$$\begin{aligned}\psi(\rho(z,gu))\leqslant&\psi\left(\max\left\{\rho(z,z),\rho(z,z),\rho(z,gu),\frac{\rho(z,gu)+\rho(z,z)}{3}\right\}\right)-\\&\phi\left(\max\left\{\rho(z,z),\rho(z,z),\rho(z,gu),\frac{\rho(z,gu)+\rho(z,z)}{3}\right\}\right)=\\&\psi(\rho(z,gu))-\phi(\rho(z,gu))\end{aligned}\tag{21}$$

这就得到 $\phi(\rho(z,gu))=0$，以及 $\rho(z,gu)=0$ 和 $z=gu$. 从而有 $z=gu=Ru$，所以(f,R)有重合点 z. 同理还可以证明 z 是(g,S)的重合点，所以映象对(f,S)和(g,R)有重合点.

步骤 4　证明映象对(f,S)和(g,R)有唯一重合点，且 f,g,R,S 有唯一公共不动点.

假设存在两个不同的重合点 $z,w(z\neq w)$，也就是 $fu=gu=Su=Ru=z$，$fv=gv=Sv=Rv=w$. 那

么由(1)式，可得

$$\psi(\rho(z, w)) = \psi(\rho(fu, gv)) \leqslant \psi(M(u, v)) - \phi(M(u, v)) \tag{22}$$

其中

$$M(u, v) = \max\left\{\rho(Su, Rv), \rho(Su, fu), \rho(Rv, gv), \frac{\rho(Rg^{-1}Su, gv) + \rho(Rv, fu)}{3}\right\} = \max\left\{\rho(z, w), \rho(z, z), \rho(w, w), \frac{\rho(z, w) + \rho(w, z)}{3}\right\} = \rho(z, w) \tag{23}$$

将(23)式带入(22)式中，得到

$$\psi(\rho(z, w)) \leqslant \psi(\rho(z, w)) - \phi(\rho(z, w))$$

从而 $\phi(\rho(z, w)) = 0$. 又因为 $\phi \in \Psi$，所以 $\rho(z, w) = 0$. 由定义 1 可知 $z = w$，与 $z \neq w$ 矛盾. 所以假设不成立，映象对(f, S)和(g, R)有唯一重合点. 再由引理 1，可得 f, g, R, S 存在唯一公共不动点.

参考文献：

[1] FRÉCHET M M. Sur Quelques Points du Calcul Fonctionnel [J]. Rendiconti del Circolo Matematico di Palermo，1906，22(1)：1－74.

[2] BRANCIARI A. A Fixed Point Theorem of Banach-Caccioppoli Type on a Class of Generalized Metric Spaces [J]. Publicationes Mathematicae-Debrecen，2000，57(1/2)：31－37.

[3] MATTHEWS S G. Partial Metric Topology [J]. Annals of the New York Academy of Sciences，1994，728：183－197.

[4] SHUKLA S. Partial Rectangular Metric Spaces and Fixed Point Theorems [J]. The Scientific World Journal，2014，2014：1－7.

[5] BOYD D W，WONG J S W. On Nonlinear Contractions [J]. Proceedings of the American Mathematical Society，1969，20(2)：458－464.

[6] RHOADES B E. Some Theorems on Weakly Contractive Maps [J]. Nonlinear Analysis，2001，47(4)：2683－2693.

[7] 尹大平，杨明歌，邓　磊. 模糊度量空间中广义压缩映射的不动点的存在性 [J]. 西南大学学报(自然科学版)，2014，36(2)：92－95

[8] ARSHAD M，AHMAD J，KARAPLNAR E. Some Common Fixed Point Results in Rectangular Metric Spaces [J]. International Journal of Analysis，2013，2013：1－7.

Common Fixed Point Theorems of Weakly (ψ, ϕ) Contractive Mappings in Partial Rectangular Metric Spaces

YAO Ting[1]，DENG Lei[1]，YANG Ming-ge[2]

1. School of Mathematics and Statistics，Southwest University，Chongqing 400715，China；

2. School of Mathematics & Science，Luoyang Normal University，Luoyang Henan 471022，China

Abstract：In this paper，we introduce the improved weakly (ψ, ϕ) contractive mapping with four self-mappings to the partial rectangular metric spaces and obtain a common fixed point theorem of the weakly (ψ, ϕ) contractive mapping in the partial rectangular metric space. Our results improve and extend some related results that have been published recently.

Key words：weakly (ψ, ϕ) contractive mapping；partial rectangular metric space；common fixed point

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