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黎曼流形上 p-Laplace 算子的 Liouville 定理[①]

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摘要:主要研究非紧黎曼流形上微分不等式 $\Delta_p u + u^\sigma \leqslant 0$ 的 Liouville 定理,其中 $1 < p \leqslant 2$,$\sigma > p-1$,证明了当积分条件 $\liminf\limits_{t \searrow 0} t^{\frac{\sigma}{\sigma-p+1}} \int_1^\infty \frac{\mu(B_r)}{r^{\frac{\sigma(p+1)-(p-1)}{\sigma-p+1}+t}} \mathrm{d}r < \infty$ 成立时上面不等式不存在弱意义下的非平凡的非负解.

关 键 词:黎曼流形;体积增长;Liouville 定理;微分不等式

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设 M 为 n 维完备非紧黎曼流形,本文考虑微分不等式

$$\Delta_p u + u^\sigma \leqslant 0 \tag{1}$$

的 Liouville 定理,其中 $1 < p \leqslant 2$,$\sigma > p-1$,$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ 为 p-Laplace 算子. 文献[1]中证明了 $\mathbb{R}^n (n > 2)$ 中 $p=2$ 时(1)式没有非平凡非负解当且仅当 $1 < \sigma < \frac{n}{n-2}$,通过选择适当的试验函数,文献[2-3]中导出了 $\mathbb{R}^n$ 中更为一般的微分不等式的 Liouville 定理. 若要将文献[2-3]中的方法推广到黎曼流形上,需要 Laplace 比较定理,从而需要 Ricci 曲率满足适当条件,文献[4]中改进了文献[2-3]中的方法,只需在体积增长的条件下就可以得到流形上相应的 Liouville 定理. 文献[5]中改进了文献[4]中的结果,得到如下定理.

定理 1 设 M 为 n 维完备非紧黎曼流形,$r > 0$ 充分大,设以某定点为球心,r 为半径的测地球 B_r 的体积满足

$$\mu(B_r) \leqslant C r^{\frac{2\sigma}{\sigma-1}} \ln^{\frac{1}{\sigma-1}} r \tag{2}$$

其中 C 是常数,则(1)式在 $p=2$ 时没有非平凡非负解.

文献[5]中构造了例子说明(2)式中的常数 $\frac{2\sigma}{\sigma-1}$,$\frac{1}{\sigma-1}$ 是最优的. 受文献[6-7]中的启发,文献[8]中证明(2)式用了更弱的条件 $\liminf\limits_{t \searrow 0} t^{\frac{\sigma}{\sigma-1}} \int_1^\infty \frac{\mu(B_r)}{r^{\frac{3\sigma-1}{\sigma-1}+t}} \mathrm{d}r < \infty$ 来替代,此时定理 1 仍然成立.

本文将考虑微分不等式(1),其中 $1 < p \leqslant 2$,$\sigma > p-1$,为此我们用 $W_{\mathrm{loc}}^p(M)$ 表示满足 $f \in L_{\mathrm{loc}}^p(M)$ 以及弱梯度 $\nabla f \in L_{\mathrm{loc}}^p(M)$ 的函数 f 组成的空间,我们用 $W_c^p(M)$ 表示 $W_{\mathrm{loc}}^p(M)$ 的带紧致支撑的函数组成的子空间,容易看出 p-Laplace 算子在 $\nabla f = 0$ 的点退化,受文献[9-10]中方法的启发,我们考虑如下的

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不等式

$$\operatorname{div}(A^{p-2}\nabla u)+u^{\sigma}\leqslant 0 \tag{3}$$

其中 $\varepsilon>0$，$A=\sqrt{|\nabla u|^2+\varepsilon}$. 容易看出(3) 式的左边是严格椭圆的.

定义 1 对任意的 $\varepsilon>0$，M 上的非负函数 u 称为(1) 式的弱解，如果 $u\in W^{p}_{\mathrm{loc}}(M)$，且对任意的非负函数 $\psi\in W^{p}_{c}(M)$，如下不等式成立

$$-\int_M A^{p-2}\nabla u\cdot\nabla\psi\mathrm{d}\mu+\int_M u^{\sigma}\psi\mathrm{d}\mu\leqslant 0 \tag{4}$$

定义 2 我们称(1) 式在弱意义下没有非平凡非负正解，是指对任意的 $\varepsilon>0$，(3) 式没有非平凡非负弱解.

下面叙述这篇文章的主要结果.

定理 2 设(M,g)为 n 维完备无边非紧黎曼流形，假设

$$\liminf_{t\searrow 0} t^{\frac{\sigma}{\sigma-p+1}}\int_1^{\infty}\frac{\mu(B_r)}{r^{\frac{\sigma(p+1)-(p-1)}{\sigma-p+1}+t}}\mathrm{d}r<\infty \tag{5}$$

则当 $1<p\leqslant 2$，$\sigma>p-1$ 时，(1) 式在弱意义下没有非平凡非负正解.

为方便，本文中 C 表示仅仅依赖于 p，σ 的正常数，前后不一定相等.

1 先验估计

引理 1 (3) 式的任意弱解 u 为正，且满足 $u^{-1}\in L^{\infty}_{\mathrm{loc}}(M)$.

证 由(3) 式知 $\operatorname{div}(A^{p-2}\nabla u)\leqslant 0$，此时应用关于弱上解的强极小值原理(参见文献[11] 中定理 8.19)，经过和文献[5] 中类似的讨论，得到引理 1. 证毕.

令 $K\subset M$ 是某个固定的非空紧集，φ 是M 上带紧致支撑的 Lipschitz 函数满足：$0\leqslant\varphi\leqslant 1$，且在 K 的某个邻域内 $\varphi=1$，特别的，$\varphi\in W^{p}_{c}(M)$. 假设 s，t 满足

$$0<t<\min\left\{p-1,\ \frac{\sigma-(p-1)}{p}\right\} \tag{6}$$

$$s>\frac{p\sigma}{\sigma-(p-1)(t+1)} \tag{7}$$

这一节的主要结果是如下的先验估计.

引理 2 设 $1<p\leqslant 2$，$\sigma>p-1$，且 u 是(1) 式在弱意义下的非负解，则 u 满足

$$\left(\int_M u^{\sigma}\varphi^{s}\mathrm{d}\mu\right)^{1-\frac{(p-1)(t+1)}{p\sigma}}\leqslant C\left[t^{-1+\frac{\sigma(1-p)}{\sigma-p+1}}\int_M|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\mathrm{d}\mu\right]^{\frac{p-1}{p}}\times$$
$$\left[\int_{M\setminus K}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_M|\nabla\varphi|^{\frac{p\sigma}{\sigma-(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{\sigma-(p-1)(t+1)}{p\sigma}} \tag{8}$$

证 令 $\psi=\varphi^{s}u^{-t}$，则 $\nabla\psi=-tu^{-t-1}\varphi^{s}\nabla u+su^{-t}\varphi^{s-1}\nabla\varphi$. 因此 $\nabla\psi\in L^{p}(M)$，即 $\psi\in W^{p}_{c}(M)$. 对给定的 $\varepsilon>0$，由(1) 有

$$t\int_M A^{p-2}u^{-t-1}\varphi^{s}|\nabla u|^2\mathrm{d}\mu+\int_M u^{\sigma}\psi\mathrm{d}\mu\leqslant s\int_M A^{p-2}u^{-t}\nabla u\cdot\nabla\varphi\mathrm{d}\mu \tag{9}$$

容易验证

$$xy\leqslant\delta x^{z}+(z-1)\delta^{-\frac{1}{z-1}}z^{-\frac{1}{z-1}}y^{\frac{z}{z-1}} \tag{10}$$

对 x，y，$\delta>0$，$z>1$ 成立. 取

$$z=\frac{p}{p-1}\qquad\delta=\frac{p-1}{p}$$

有

$$xy\leqslant\left(1-\frac{1}{p}\right)x^{\frac{p}{p-1}}+\frac{1}{p}y^{p}$$

从而

$$\begin{aligned}&s\int_M A^{p-2}u^{-t}\varphi^{s-1}\nabla u\cdot\nabla\varphi\mathrm{d}\mu\leqslant\\&\int_M(A^{p-2}t^{\frac{p-1}{p}}\mid\nabla u\mid u^{-\frac{(p-1)(t+1)}{p}}\varphi^{\frac{p-1}{p}s})\cdot(t^{-\frac{p-1}{p}}su^{-t+\frac{(p-1)(t+1)}{p}}\varphi^{\frac{s}{p}-1}\mid\nabla\varphi\mid)\mathrm{d}\mu\leqslant\\&\frac{p-1}{p}\int_M(A^{p-2}t^{\frac{p-1}{p}}\mid\nabla u\mid u^{-\frac{(p-1)(t+1)}{p}}\varphi^{\frac{p-1}{p}s})^{\frac{p}{p-1}}\mathrm{d}\mu+\frac{1}{p}\int_M(t^{-\frac{p-1}{p}}su^{-t+\frac{(p-1)(t+1)}{p}}\varphi^{\frac{s}{p}-1}\mid\nabla\varphi\mid)^p\mathrm{d}\mu=\\&\frac{p-1}{p}t\int_M A^{\frac{p(p-2)}{p-1}}\mid\nabla u\mid^{\frac{p}{p-1}}u^{-t-1}\varphi^{s}\mathrm{d}\mu+\frac{1}{p}t^{-(p-1)}s^p\int_M u^{-t+(p-1)}\varphi^{s-p}\mid\nabla\varphi\mid^p\mathrm{d}\mu\end{aligned}$$

注意到 $A^{\frac{p(p-2)}{p-1}}\mid\nabla u\mid^{\frac{p}{p-1}}\leqslant A^{p-2}\mid\nabla u\mid^2$ 对 $1<p\leqslant 2$ 成立，于是

$$\begin{aligned}&s\int_M A^{p-2}u^{-t}\varphi^{s-1}\nabla u\cdot\nabla\varphi\mathrm{d}\mu\leqslant\\&\frac{p-1}{p}t\int_M A^{p-2}\mid\nabla u\mid^2u^{-t-1}\varphi^{s}\mathrm{d}\mu+\frac{1}{p}t^{-(p-1)}s^p\int_M u^{-t+(p-1)}\varphi^{s-p}\mid\nabla\varphi\mid^p\mathrm{d}\mu\end{aligned}\tag{11}$$

将此式代入(9) 式得

$$\frac{t}{p}\int_M A^{p-2}u^{-t-1}\varphi^{s}\mid\nabla u\mid^2\mathrm{d}\mu+\int_M u^{\sigma-t}\varphi^{s}\mathrm{d}\mu\leqslant\frac{1}{p}t^{-(p-1)}s^p\int_M u^{-t+(p-1)}\varphi^{s-p}\mid\nabla\varphi\mid^p\mathrm{d}\mu\tag{12}$$

再将(10) 式中取 $\delta=1-\dfrac{\sigma-p+1}{\sigma-t}$，$z=\dfrac{\sigma-t}{-t+p-1}$ 有

$$xy\leqslant\left(1-\frac{\sigma-p+1}{\sigma-t}\right)x^{\frac{\sigma-p+1}{\sigma-t}}+\frac{\sigma-p+1}{\sigma-t}y^{\frac{\sigma-t}{\sigma-p+1}}$$

因此

$$\begin{aligned}&\frac{1}{p}t^{-(p-1)}s^p\int_M u^{-t+(p-1)}\varphi^{s-p}\mid\nabla\varphi\mid^p\mathrm{d}\mu=\\&\int_M(u^{-t+(p-1)}\varphi^{\frac{s(-t+p-1)}{\sigma-t}})\cdot(p^{-1}t^{-(p-1)}s^p\varphi^{s-p-\frac{s(-t+p-1)}{\sigma-t}}\mid\nabla\varphi\mid^p)\mathrm{d}\mu\leqslant\\&\int_M(u^{-t+(p-1)}\varphi^{\frac{s(-t+p-1)}{\sigma-t}})^{\frac{\sigma-t}{-t+p-1}}\mathrm{d}\mu+\frac{\sigma-p+1}{\sigma-t}\int_M(p^{-1}t^{-(p-1)}s^p\varphi^{s-p-\frac{s(-t+p-1)}{\sigma-t}}\mid\nabla\varphi\mid^p)^{\frac{\sigma-t}{\sigma-p+1}}\mathrm{d}\mu=\\&\left(1-\frac{\sigma-p+1}{\sigma-t}\right)\int_M u^{\sigma-t}\varphi^{s}\mathrm{d}\mu+\frac{\sigma-p+1}{\sigma-t}p^{\frac{t-\sigma}{\sigma-p+1}}t^{\frac{(\sigma-t)(1-p)}{\sigma-p+1}}s^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_M\varphi^{s-\frac{\sigma-t}{\sigma-p+1}}\mid\nabla\varphi\mid^{\frac{\sigma-t}{\sigma-p+1}}\mathrm{d}\mu\end{aligned}$$

将此式代入(12) 得

$$\begin{aligned}&\frac{t}{p}\int_M A^{p-2}u^{-t-1}\varphi^{s}\mid\nabla u\mid^2\mathrm{d}\mu+\frac{\sigma-p+1}{\sigma-t}\int_M u^{\sigma-t}\varphi^{s}\mathrm{d}\mu\leqslant\\&Ct^{\frac{(\sigma-t)(1-p)}{\sigma-p+1}}\int_M\varphi^{s-\frac{\sigma-t}{\sigma-p+1}}\mid\nabla\varphi\mid^{\frac{p(\sigma-t)}{\sigma-p+1}}\mathrm{d}\mu\leqslant Ct^{\frac{\sigma(1-p)}{\sigma-p+1}}\int_M\mid\nabla\varphi\mid^{\frac{p(\sigma-t)}{\sigma-p+1}}\mathrm{d}\mu\end{aligned}\tag{13}$$

这里最后一个不等式归因于 $s>\dfrac{\sigma-t}{\sigma-p+1}$，这个不等式可以由(7) 式推出. 现在我们在(4) 式中取 $\psi=\varphi^s$ 得

$$-\int_M A^{p-2}\nabla u\cdot\nabla\varphi^s\mathrm{d}\mu+\int_M u^{\sigma}\varphi^{s}\mathrm{d}\mu\leqslant 0$$

因此

$$\int_M u^{\sigma}\varphi^{s}\mathrm{d}\mu\leqslant s\int_M A^{p-2}\varphi^{s-1}\nabla u\cdot\nabla\varphi\mathrm{d}\mu\leqslant$$

$$s\left[\int_M A^{\frac{p(p-2)}{p-1}}|\nabla u|^{\frac{p}{p-1}}\varphi^s u^{-t-1}\mathrm{d}\mu\right]^{\frac{p-1}{p}}\cdot\int_M\varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\mathrm{d}\mu\Big]^{\frac{1}{p}}\leqslant$$

$$s\left[\int_M A^{p-2}|\nabla u|^2\varphi^s u^{-t-1}\mathrm{d}\mu\right]^{\frac{p-1}{p}}\cdot\left[\int_M\varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\mathrm{d}\mu\right]^{\frac{1}{p}} \tag{14}$$

由(13)式我们有

$$\int_M A^{p-2}u^{-t-1}\varphi^s|\nabla\varphi|^2\mathrm{d}\mu\leqslant Ct^{-1+\frac{\sigma(1-p)}{\sigma-p+1}}\int_M|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\mathrm{d}\mu$$

将这个不等式代入(14)式得

$$\int_M u^\sigma\varphi^s\mathrm{d}\mu\leqslant C\left[t^{-1+\frac{\sigma(1-p)}{\sigma-p+1}}\int_M|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\mathrm{d}\mu\right]^{\frac{p-1}{p}}\cdot\left[\int_M\varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\mathrm{d}\mu\right]^{\frac{1}{p}} \tag{15}$$

对指标对 $\dfrac{\sigma}{(p-1)(t+1)}$, $\dfrac{\sigma}{\sigma-(p-1)(t+1)}$ 应用 Hölder 不等式有

$$\int_M\varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\mathrm{d}\mu=\int_{M\backslash K}\varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\mathrm{d}\mu\leqslant$$

$$\left[\int_{M\backslash K}\left(u^{(p-1)(t+1)}\varphi^{\frac{(p-1)(t+1)s}{\sigma}}\right)^{\frac{\sigma}{(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{\sigma}}\cdot$$

$$\left[\int_{M\backslash K}\left(\varphi^{s-p-\frac{(p-1)(t+1)s}{\sigma}}|\nabla\varphi|^p\right)^{\frac{\sigma}{\sigma-(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{\sigma-(p-1)(t+1)}{\sigma}}=$$

$$\left[\int_{M\backslash K}u^\sigma\varphi^s\mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{\sigma}}\cdot\left[\int_{M\backslash K}\varphi^{s-\frac{p\sigma}{\sigma-(p-1)(t+1)}}|\nabla\varphi|^{\frac{p\sigma}{\sigma-(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{\sigma-(p-1)(t+1)}{\sigma}}\leqslant$$

$$\left[\int_{M\backslash K}u^\sigma\varphi^s\mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{\sigma}}\cdot\left[\int_{M\backslash K}|\nabla\varphi|^{\frac{p\sigma}{\sigma-(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{\sigma-(p-1)(t+1)}{\sigma}}$$

这里最后一个不等式归因于(7)式, $0\leqslant\varphi\leqslant 1$ 以及 φ 在 K 中有紧致支撑. 将上式代入(15)式得

$$\int_M u^\sigma\varphi^s\mathrm{d}\mu\leqslant C\left[t^{-1+\frac{\sigma(1-p)}{\sigma-p+1}}\int_M|\nabla\varphi|^{\frac{p(\sigma-1)}{\sigma-(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{p-1}{p}}\cdot$$

$$\left[\int_{M\backslash K}u^\sigma\varphi^s\mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{\sigma}}\cdot\left[\int_{M\backslash K}|\nabla\varphi|^{\frac{p\sigma}{\sigma-(p-1)(t+1)}}\mathrm{d}\mu\right]^{\frac{\sigma-(p-1)(t+1)}{p\sigma}} \tag{16}$$

由于 $A^{\frac{p(p-2)}{p-1}}|\nabla u|^{\frac{p}{p-1}}\leqslant A^{p-2}|\nabla u|^2$, $0\leqslant\varphi\leqslant 1$, $\varphi\in W_c^p(M)$ 我们有

$$\left|\int_M A^{p-2}\nabla u\cdot\nabla\varphi^s\mathrm{d}\mu\right|\leqslant\left(\int_{\mathrm{supp}(\varphi)}|A^{p-2}\nabla u|^{\frac{p}{p-1}}\mathrm{d}\mu\right)^{\frac{p-1}{p}}\left(\int_M|\nabla\varphi^s|^p\mathrm{d}\mu\right)\frac{1}{p}\leqslant$$

$$\left(\int_{\mathrm{supp}(\varphi)}|\nabla u|^p\mathrm{d}\mu\right)^{\frac{p-1}{p}}\left(\int_M|\nabla\varphi^s|^p\mathrm{d}\mu\right)\frac{1}{p}\leqslant C\left(\int_{\mathrm{supp}(\varphi)}|\nabla u|^p\mathrm{d}\mu\right)^{\frac{p-1}{p}}\left(\int_M|\nabla\varphi|^p\mathrm{d}\mu\right)\frac{1}{p}<\infty$$

这里 $\mathrm{supp}(\varphi)$ 表示 φ 的紧致支撑集. 此估计结合

$$-\int_M A^{p-2}\nabla u\cdot\nabla\varphi^s\mathrm{d}\mu+\int_M u^\sigma\varphi^s\mathrm{d}\mu\leqslant 0$$

表明 $\int_M u^\sigma\varphi^s\mathrm{d}\mu$ 是有界的, 于是由(16)式得(8)式. 证毕.

2 定理2的证明

假设 $R>0$ 充分大使得 $t=\dfrac{1}{\ln R}$ 满足(6)式. 令

$$\xi_n(x)=\begin{cases}1 & 0\leqslant r(x)\leqslant 2^nR\\ 2-\dfrac{r(x)}{2^nR} & 2^nR\leqslant r(x)\leqslant 2^{n+1}R\\ 0 & r(x)\geqslant 2^{n+1}R\end{cases}$$

定义

$$J_n(a):=\int_M|\nabla\psi_n|^a\mathrm{d}\mu$$

其中

$$\psi_n(x)=\varphi(x)\xi_n(x)$$

且 φ 定义如下

$$\varphi(x)=\begin{cases}1 & r(x)<R\\ \left(\dfrac{r(x)}{R}\right)^{-t} & r(x)\geqslant R\end{cases}$$

下面给出 $J_n(a)$ 的一个关键估计，这里所用的方法类似于文献[8] 中的方法.

引理 3　我们假设 $a\in\left[1,\dfrac{2p\sigma}{\sigma-(p-1)}\right]$满足

$$a(t+1)\geqslant t+\frac{p\sigma}{\sigma-(p-1)}\tag{17}$$

则

$$\limsup_{n\to\infty}J_n(a)\leqslant Ct^aM(t)\tag{18}$$

其中

$$M(t)=\int_1^\infty\frac{\mu(B_r)}{r^{\frac{\sigma(p+1)-(p-1)}{\sigma-p+1}+t}}\mathrm{d}r$$

证　类似于文献[5,8]，对 $a\geqslant1$，有

$$J_n(a)\leqslant C\left[\int_M\xi_n^a|\nabla\varphi|^a\mathrm{d}\mu+\int_M\varphi_n^a|\nabla\xi_n|^a\mathrm{d}\mu\right]\leqslant$$
$$C\left[\int_{M\backslash B_R}\xi_n^a|\nabla\varphi|^a\mathrm{d}\mu+\int_{B_{2^{n+1}R}\backslash B_{2^nR}}\varphi_n^a|\nabla\xi_n|^a\mathrm{d}\mu\right]:=C(I(a)+II_n(a))$$

由(20) 式，类似于[8] 中的讨论知

$$I(a)\leqslant Ct^a\sum_{n=1}^\infty\frac{V(2^nR)}{(2^{n-1}R)^{a(t+1)}}\leqslant Ct^aM(t)$$

并且当 $n\to\infty$ 时，有

$$II_n(a)\leqslant C\frac{V(2^{n+1}R)}{(2^nR)^{a(t+1)}}\longrightarrow0$$

因此(18) 式成立. 证毕.

现在我们来证明定理 2.

证　取 $a=\dfrac{p(\sigma-t)}{\sigma-p+1}$ 和 $a=\dfrac{p\sigma}{\sigma-(p-1)(t+1)}$，易知都有 $a\in\left[1,\dfrac{2p\sigma}{\sigma-p+1}\right]$且 a 满足(17) 式，由引理 3 知

$$\limsup_{n\to\infty}J_n\left(\frac{p(\sigma-t)}{\sigma-p+1}\right)\leqslant Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}M(t)\tag{19}$$

$$\limsup_{n\to\infty}J_n\left(\frac{p\sigma}{\sigma-(p-1)(t+1)}\right)\leqslant Ct^{\frac{p\sigma}{\sigma-(p-1)(t+1)}}M(t)\tag{20}$$

由(8) 式有

$$\left(\int_M u^\sigma \psi_n^s \mathrm{d}\mu\right)^{1-\frac{(p-1)(t+1)}{p\sigma}} \leqslant C\left[t^{-1+\frac{\sigma(1-p)}{\sigma-p+1}} J_n\left(\frac{p(\sigma-t)}{\sigma-p+1}\right)\right]^{\frac{p-1}{p}} \cdot \left[J_n\left(\frac{p\sigma}{\sigma-(p-1)(t+1)}\right)\right]^{\frac{\sigma-(p-1)(t+1)}{p\sigma}}$$

令 $n\to\infty$ 并利用(19)式,(20)式得

$$\left(\int_M u^\sigma \psi^s \mathrm{d}\mu\right)^{1-\frac{(p-1)(t+1)}{p\sigma}} \leqslant Ct^\Theta (M(t))^{1-\frac{(p-1)(t+1)}{p\sigma}}$$

其中

$$\Theta=\left(-1+\frac{\sigma(1-p)}{\sigma-p+1}\right)\cdot\frac{p-1}{p}+\frac{p(\sigma-t)}{\sigma-p+1}\cdot\frac{p-1}{p}+1=\frac{\sigma p-p+1-p^2t+pt}{p(\sigma-p+1)}$$

因此

$$\int_M u^\sigma \psi^s \mathrm{d}\mu \leqslant C^{\frac{p\sigma}{p\sigma-(p-1)(t+1)}} t^{\frac{\sigma(p\sigma-(p-1)(pt+1))}{(\sigma-p+1)(p\sigma-(p-1)(t+1))}} M(t)$$

由(5)式知对充分小的 $t>0$,$C^{\frac{p\sigma}{p\sigma-(p-1)(t+1)}} t^{\frac{\sigma}{\sigma-p+1}} M(t)\leqslant C$,注意到

$$\frac{\sigma(p\sigma-(p-1)(pt+1))}{(\sigma-p+1)(p\sigma-(p-1)(t+1))}-\frac{\sigma}{\sigma-p+1}=-\frac{\sigma(p-1)^2 t}{(\sigma-p+1)(p\sigma-(p-1)(t+1))}$$

因此

$$\int_M u^\sigma \psi^s \mathrm{d}\mu \leqslant C$$

即

$$\int_{B_R} u^\sigma \mathrm{d}\mu \leqslant C$$

令 $R\to\infty$ 得 $\int_M u^\sigma \mathrm{d}\mu \leqslant C$. 由不等式(8)式知

$$\int_M u^\sigma \psi_n^s \mathrm{d}\mu \leqslant$$

$$C\left[\int_{M\backslash B_R} u^\sigma \psi_n^s \mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{p\sigma}} \cdot \left[t^{-1+\frac{\sigma(1-p)}{\sigma-p+1}} J_n\left(\frac{p(\sigma-t)}{\sigma-p+1}\right)\right]^{\frac{p-1}{p}} \cdot \left[J_n\left(\frac{p\sigma}{\sigma-(p-1)(t+1)}\right)\right]^{\frac{\sigma-(p-1)(t+1)}{p\sigma}}$$

令 $n\to\infty$ 得

$$\int_M u^\sigma \psi^s \mathrm{d}\mu \leqslant C\left[\int_{M\backslash B_R} u^\sigma \psi^s \mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{p\sigma}}$$

则

$$\int_{B_R} u^\sigma \mathrm{d}\mu \leqslant C\left[\int_{M\backslash B_R} u^\sigma \mathrm{d}\mu\right]^{\frac{(p-1)(t+1)}{p\sigma}} \tag{21}$$

由 $\int_M u^\sigma \mathrm{d}\mu \leqslant C$ 知 $R\to\infty$ 时,$\int_{M\backslash B_R} u^\sigma \mathrm{d}\mu \to 0$. 在(21)式中令 $R\to\infty$ 得 $\int_M u^\sigma \mathrm{d}\mu=0$. 证毕.

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A Liouville Theorem of the p-Laplace Operator on Riemannian Manifolds

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Abstract: In this paper we study the Liouville theorem of the differential inequality $\Delta_p u+u^{\sigma}\leqslant 0$ on complete noncompact Riemannian manifolds, where $1<p\leqslant 2$, $\sigma>p-1$. We prove that the above inequality does not exist nontrivial nonnegative solutions in the weak sense if the integral condition $\liminf_{t\searrow 0} t^{\frac{\sigma}{\sigma-p+1}}\int_1^{\infty}\frac{\mu(B_r)}{r^{\frac{\sigma(p+1)-(p-1)}{\sigma-p+1}+t}}\mathrm{d}r<\infty$ satisfies.

Key words: Riemannian manifold; volume growth; Liouville theorem; differential inequality

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