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薄域上非自治随机反应扩散方程吸引子的存在性[①]

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摘要:主要研究薄域上带加法噪音的反应扩散方程的极限行为,证明了在 $n+1$ 维薄域上该方程的拉回吸引子的存在性、唯一性.

关 键 词:薄域;随机反应扩散方程;加法噪音;拉回吸引子

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目前已有很多文献从不同的角度对薄域问题进行了大量的研究. 文献[1] 研究了薄域上带乘法噪音的非自治反应扩散方程的极限行为,证明了拉回吸引子的存在性、唯一性,但未对带加法噪音的反应扩散方程进行讨论. 因此本文将主要研究 $n+1$ 维薄域上带加法噪音的反应扩散方程的极限行为,研究其拉回吸引子的存在性、唯一性. 首先,我们将随机反应扩散方程从扰动薄域上转化到固定区域上,并证明了在状态空间上存在一个非自治动力系统. 然后对方程的解做一致性估计. 最后给出本文的主要结论,证明了 $\mathscr{D}_1$-拉回吸引子 $\mathscr{A}_\varepsilon=\{\mathscr{A}_\varepsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathscr{D}_1$ 的存在性、唯一性,并且证明了当 $n+1$ 维薄域降为 n 维区域时,该方程也具有 $\mathscr{D}_0$-拉回吸引子 $\mathscr{A}_0=\{\mathscr{A}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathscr{D}_0$.

1 非自治随机反应扩散方程的协循环

设 $\mathscr{Q}$ 是 $\mathbb{R}^n$ 中的光滑有界区域,$\mathscr{O}_\varepsilon$ 是定义如下的 $n+1$ 维区域:

$$\mathscr{O}_\varepsilon=\{x=(x^*,x_{n+1}):x^*=(x_1,\cdots,x_n)\in\mathscr{Q},0<x_{n+1}<\varepsilon g(x^*)\}$$

其中 $g\in C^2(\overline{\mathscr{Q}},(0,+\infty))$,$0<\varepsilon\leqslant 1$. 易知存在正数 r_1 和 r_2,使得对 $\forall x^*\in\overline{\mathscr{Q}}$,有 $r_1\leqslant g(x^*)\leqslant r_2$. 令 $\mathscr{O}=\mathscr{Q}\times(0,1)$,$\widetilde{\mathscr{O}}=\mathscr{Q}\times(0,r_2)$. 易知对 $\forall\varepsilon\in(0,1)$,有 $\mathscr{O}_\varepsilon\in\widetilde{\mathscr{O}}$. 任意给定 $\tau\in\mathbb{R}$,考虑下面的方程:

$$\begin{cases}\mathrm{d}\hat{u}^\varepsilon-\Delta\hat{u}^\varepsilon\mathrm{d}t+\lambda\hat{u}^\varepsilon\mathrm{d}t=(f(t,x,\hat{u}^\varepsilon)+G(t,x))\mathrm{d}t+h(x)\circ\mathrm{d}W & x\in\mathscr{O}_\varepsilon,\ t>\tau\\ \dfrac{\partial\hat{u}^\varepsilon}{\partial\nu_\varepsilon}=0 & x\in\partial\mathscr{O}_\varepsilon\end{cases}\tag{1}$$

初值为

$$\hat{u}^\varepsilon(\tau,x)=\hat{u}^\varepsilon_\tau(x)\qquad x\in\mathscr{O}_\varepsilon\tag{2}$$

其中 $\lambda>0$,ν_ε 是边界 $\partial\mathscr{O}_\varepsilon$ 的单位外法向量,$G\in L^2_{\mathrm{loc}}(\mathbb{R},L^\infty(\widetilde{\mathscr{O}}))$,$h(x)\in H^2(\widetilde{\mathscr{O}})\cap W^{2,+\infty}(\widetilde{\mathscr{O}})$,$W$ 是定义在概率空间的双边实值的 Wiener 过程,f 是非线性函数,并且满足下列条件:对 $\forall x\in\widetilde{\mathscr{O}}$,$\forall t,s\in\mathbb{R}$,有:

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$$f(t,x,s)s \leqslant -\alpha_1 |s|^p + \psi_1(t,x) \tag{3}$$

$$|f(t,x,s)| \leqslant \alpha_2 |s|^{p-1} + \psi_2(t,x) \tag{4}$$

$$\frac{\partial f(t,x,s)}{\partial s} \leqslant \beta \tag{5}$$

$$\left|\frac{\partial f(t,x,s)}{\partial x}\right| \leqslant \psi_3(t,x) \tag{6}$$

其中 $p \geqslant 2$，α_1, α_2 和 β 是正常数，$\psi_1 \in L^1_{\mathrm{loc}}(\mathbb{R}, L^\infty(\tilde{\mathcal{O}}))$，$\psi_2, \psi_3 \in L^2_{\mathrm{loc}}(\mathbb{R}, L^\infty(\tilde{\mathcal{O}}))$.

当 $\varepsilon \to 0$ 时，$n+1$ 维区域 $\mathcal{O}_\varepsilon$ 降为 n 维区域 $\mathcal{Q}$，那么当 $\varepsilon = 0$ 时，方程(1)-(2)变为

$$\begin{cases} \mathrm{d}u^0 - \dfrac{1}{g}\sum\limits_{i=1}^{n}(gu^0_{y_i})_{y_i}\mathrm{d}t + \lambda u^0\mathrm{d}t = (f(t,y^*,0,u^0) + G(t,y^*,0))\mathrm{d}t + h(y^*,0)\circ \mathrm{d}W \\ y^* = (y_1,\cdots,y_n) \in \mathcal{Q} \qquad t > \tau \\ \dfrac{\partial \hat{u}^0}{\partial \nu_0} = 0 \qquad y^* \in \partial\mathcal{Q} \end{cases} \tag{7}$$

初值为

$$u^0(\tau, y^*) = u^0_\tau(y^*) \qquad y^* \in \mathcal{Q} \tag{8}$$

下面将方程(1)-(2)转化为区域 $\mathcal{O}$ 上的有界边界值问题. 对 $\forall x = (x^*, x_{n+1})$，定义变换：

$$T_\varepsilon: \mathcal{O}_\varepsilon \longrightarrow \mathcal{O}$$

$$T_\varepsilon(x^*, x_{n+1}) = \left(x^*, \frac{x_{n+1}}{\varepsilon g(x^*)}\right)$$

再令 $y = (y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1})$，则有：

$$x^* = y^* \qquad x_{n+1} = \varepsilon g(y^*) y_{n+1}$$

设 $\boldsymbol{J}$ 为 T_ε 的雅可比矩阵，计算易得 $\boldsymbol{J}$ 的行列式为 $|\boldsymbol{J}| = \dfrac{1}{\varepsilon g(y^*)}$. 令 $\boldsymbol{J}^{\mathrm{T}}$ 为 $\boldsymbol{J}$ 的转置矩阵. 由文献[2]可知，梯度算子 ∇、Laplace 算子 Δ、散度算子 div 在初始变量 $x \in \mathcal{O}_\varepsilon$ 和新的变量 $y \in \mathcal{O}$ 下，有：

$$\nabla_x \hat{u}(x) = \boldsymbol{J}^{\mathrm{T}} \nabla_y u(y) \qquad \Delta_x \hat{u}(x) = \frac{1}{g}\mathrm{div}_y(P_\varepsilon u(y))$$

其中 $u(y) = \hat{u}(x)$，P_ε 是如下定义的算子：

$$P_\varepsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_{n+1} u_{y_{n+1}} \\ \vdots \\ g u_{y_n} - g_{y_n} y_{n+1} u_{y_{n+1}} \\ -\sum\limits_{i=1}^{n}\left[y_{n+1} g_{y_i} u_{y_i} + \dfrac{1}{\varepsilon^2 g}\left(1 + \sum\limits_{i=1}^{n}(\varepsilon y_{n+1} g_{y_i})^2\right) u_{y_{n+1}}\right] \end{pmatrix}$$

对 $x = (x^*, x_{n+1})$，将 $f(t,x,s), G(t,x)$ 改写为 $f(t,x^*,x_{n+1},s), G(t,x^*,x_{n+1})$. 另外，对 $y = (y^*, y_{n+1}) \in \mathcal{O}$ 和 $t, s \in \mathbb{R}$，约定：

$$G_\varepsilon(t,y^*,y_{n+1}) = G(t,y^*,\varepsilon g(y^*)y_{n+1}) \qquad G_0(t,y^*) = G(t,y^*,0)$$

$$f_\varepsilon(t,y^*,y_{n+1},s) = f(t,y^*,\varepsilon g(y^*)y_{n+1},S) \qquad f_0(t,y^*,s) = f(t,y^*,0,s)$$

$$h_\varepsilon(y^*,y_{n+1}) = h(y^*,\varepsilon g(y^*)y_{n+1}) \qquad h_0(y^*) = h(y^*,0)$$

对于上述薄域空间，引进新的函数空间并定义相关内积与范数. 首先，赋予空间 $L^2(\mathcal{O})$ 内积 $(u,v)_{H_g(\mathcal{O})} = \int_{\mathcal{O}} guv\mathrm{d}y$，记作 $H_g(\mathcal{O})$. 易知 $H_g(\mathcal{O})$ 是希尔伯特空间，且其范数等价于 $L^2(\mathcal{O})$ 的经典范数. 对 $0 < \varepsilon \leqslant 1$，定义双线性形式 $a_\varepsilon(\cdot,\cdot)$：

$$a_\varepsilon(u,v) = (\boldsymbol{J}^{\mathrm{T}}\nabla_y u, \boldsymbol{J}^{\mathrm{T}}\nabla_y v)_{H_g(\mathcal{O})} \qquad \forall v, u \in H^1(\mathcal{O})$$

其中

$$\boldsymbol{J}^{\mathrm{T}}\nabla_y u(y)=\left(u_{y_1}-\frac{g_{y_1}}{g}y_{n+1}u_{y_{n+1}},\ \cdots,\ u_{y_n}-\frac{g_{y_n}}{g}y_{n+1}u_{y_{n+1}},\ \frac{1}{\varepsilon g}u_{y_{n+1}}\right)$$

赋予空间 $H^1(\mathscr{O})$ 范数 $\|u\|_{H^1_\varepsilon(\mathscr{O})}=(a_\varepsilon(u,u)+\|u\|^2_{L^2(\mathscr{O})})^{\frac{1}{2}}$，并记作 $H^1_\varepsilon\mathscr{O}$. 定义算子 $A_\varepsilon u=-\frac{1}{g}\operatorname{div}_y(P_\varepsilon u)$，定义域为

$$D(A_\varepsilon)=\{u\in H^2(\mathscr{O})\cap H_g(\mathscr{O}):P_\varepsilon u\cdot\nu=0,\ y\in\partial\mathscr{O}\}$$

则有

$$a_\varepsilon(u,v)=(A_\varepsilon u,v)_{H_g(\mathscr{O})}\qquad\forall u\in D(A_\varepsilon),\ \forall v\in H^1(\mathscr{O})$$

利用算子 A_ε，由文献[1]，方程(1)－(2) 可变换为

$$\begin{cases}\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}t}+A_\varepsilon u^\varepsilon+\lambda u^\varepsilon=f_\varepsilon(t,y,u^\varepsilon)+G_\varepsilon(t,y)+h_\varepsilon(y)\circ\dfrac{\mathrm{d}W}{\mathrm{d}t}\\ u^\varepsilon(\tau)=u^\varepsilon_\tau\end{cases}\tag{9}$$

同理，为了对方程(7)－(8) 做处理，赋予 $L^2(\mathscr{Q})$ 内积 $(u,v)_{H_g(\mathscr{Q})}=\int_{\mathscr{Q}}guv\mathrm{d}y^*\ (\forall u,v\in L^2(\mathscr{Q}))$，并记作 $H_g(\mathscr{Q})$. 定义双线性形式

$$a_0(u,v)=\int_{\mathscr{Q}}g\nabla u\cdot\nabla v\mathrm{d}y^*\qquad\forall u,v\in H^1(\mathscr{Q})$$

最后定义算子 $A_0=-\frac{1}{g}\sum_{i=1}^{n}(gu_{y_i})_{y_i}$，定义域为 $D(A_0)=\left\{u\in H^2(\mathscr{Q})\cap H_g(\mathscr{Q}):\frac{\partial u}{\partial\nu_0}=0,\ y\in\partial\mathscr{Q}\right\}$. 由算子 A_0，方程(7)－(8) 可转化为

$$\begin{cases}\dfrac{\mathrm{d}u^0}{\mathrm{d}t}+A_0u^0+\lambda u^0=f_0(t,y^*,u^0)+G_0(t,y^*)+h_0(y^*)\circ\dfrac{\mathrm{d}W}{\mathrm{d}t}\qquad y^*\in\mathscr{Q},\ t>\tau\\ u^0(\tau)=u^0_\tau\end{cases}\tag{10}$$

下面考虑概率空间$(\Omega,\mathscr{F},P)$，其中$\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}$，$\mathscr{F}$是由$\Omega$ 的紧开拓扑导出的 Borel 代数，P 是$(\Omega,\mathscr{F})$ 上对应的 Wiener 测度. 定义群 θ_t：

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)\qquad\omega\in\Omega,\ t\in\mathbb{R}$$

那么$(\Omega,\mathscr{F},P,\{\theta_t\}_{t\in\mathbb{R}})$ 是度量动力系统. 下面借助$(\Omega,\mathscr{F},P,\{\theta_t\}_{t\in\mathbb{R}})$ 来建立方程(9) 的连续协循环. 首先，任给定 $\omega\in\Omega$，令

$$z(\omega)=-\lambda\int_{-\infty}^{0}\mathrm{e}^{\lambda\tau}\omega(\tau)\mathrm{d}\tau$$

易知$z(\omega)$是方程 $\mathrm{d}z+\lambda z\mathrm{d}t=\mathrm{d}\omega$ 的稳定解. 存在θ-不变且P-可测的子集$\widetilde{\Omega}\in\Omega$，使得对 $\forall\omega\in\Omega$，$z(\theta_t\omega)$ 关于 t 是连续的，并且 $|z(\theta_t\omega)|$ 是 Tempered(详见文献[3－5]). 方便起见，我们仍把 $\widetilde{\Omega}$ 记作Ω. 令

$$v^\varepsilon(t)=u^\varepsilon(t)-h_\varepsilon z(\theta_t\omega)$$

那么 v^ε 应满足

$$\begin{cases}\dfrac{\mathrm{d}v^\varepsilon}{\mathrm{d}t}+A_\varepsilon v^\varepsilon+\lambda v^\varepsilon=f_\varepsilon(t,y,v^\varepsilon+h_\varepsilon z(\theta_t\omega))+G_\varepsilon(t,y)-z(\theta_t\omega)A_\varepsilon(h_\varepsilon)\\ v^\varepsilon(\tau,\tau,\omega,v^\varepsilon_\tau)=v^\varepsilon_\tau\end{cases}\tag{11}$$

因为方程(11) 是含有参数$\omega\in\Omega$ 的确定方程，由文献[6] 可知，如果 f 满足(3)－(6) 式，那么对 $\forall\omega\in\Omega$，$\tau\in\mathbb{R}$ 和 $v^\varepsilon_\tau\in L^2(\mathscr{O})$，$\forall T>0$，方程(11) 拥有唯一的解 $v^\varepsilon(\cdot,\tau,\omega,v^\varepsilon_\tau)\in C([\tau,\infty),L^2(\mathscr{O}))\cap L^2((\tau,\tau+T),H^1(\mathscr{O}))$. 另外，此解是$(\mathscr{F},\mathscr{B}(L^2(\mathscr{O})))$ 可测的，且关于初值 v^ε_τ 连续. 定义方程(9) 的协循环 Φ_ε：$\mathbb{R}_+\times\mathbb{R}\times\Omega\times L^2(\mathscr{O})\longrightarrow L^2(\mathscr{O})$，对 $\forall t\in\mathbb{R}_+$，$\omega\in\Omega$ 和 u^ε_τ，令

$$\Phi_\varepsilon(t,\tau,\omega,u^\varepsilon_\tau)=u^\varepsilon(t+\tau,\tau,\theta_{-\tau}\omega,u^\varepsilon_\tau)=v^\varepsilon(t+\tau,\tau,\theta_{-\tau}\omega,v^\varepsilon_\tau)+h_\varepsilon z(\theta_t\omega)\tag{12}$$

其中 $v_\tau^\varepsilon = u_\tau^\varepsilon - h_\varepsilon z(\theta_\tau\omega)$. 由文献[7]可知映射 Φ_ε 是 $L^2(\mathcal{O})$ 上关于 $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ 的连续协循环. 因此,在本文中,我们只考虑非自治随机动力系统 Φ_ε.

相似的变换 $v^0(t, \tau, \omega, v_\tau^0) = u^0(t, \tau, \omega, u_\tau^0) - h_0 z(\theta_t\omega)$ 可以将方程(10)转化为如下方程:

$$\begin{cases}\dfrac{\mathrm{d}v^0}{\mathrm{d}t} + A_0 v^0 + \lambda v^0 = f_0(t, y, v^0 + h_0 z(\theta_t\omega)) + G_0(t, y^*) - A_0(h_0 z(\theta_t\omega)) & y^* \in \mathcal{Q}, t \geqslant \tau \\ v^0(\tau, \tau, \omega, v_\tau^0) = v_\tau^0 \end{cases} \tag{13}$$

类似地,可证得方程(10)在 $L^2(\mathcal{Q})$ 上存在连续协循环 $\Phi_0(t, \tau, \omega, u_\tau^0)$.

记 $X_\varepsilon = L^2(\mathcal{O}_\varepsilon)$, $X_0 = L^2(\mathcal{Q})$, $X_1 = L^2(\mathcal{O})$. 对 $i = \varepsilon, 0, 1$, 令 $B_i = \{B_i(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\}$ 为 X_i 的非空子集族. 如果对 $\forall c > 0$, 有

$$\lim_{t\to-\infty} \mathrm{e}^{ct} \| B_i(\tau + t, \theta_t\omega) \|_{X_i} = 0$$

其中 $\| B_i \|_{X_i} = \sup_{x\in B_i} \| x \|_{X_i}$, 则称 B_i 是 Tempered. 再令

$$\mathcal{D}_i = \{B_i = \{B_i(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\}: B_i \text{ 在 } X_i \text{ 上是 Tempered}\}$$

为了得到解的一致估计,我们对外力项做如下假设:

$$\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\| G(s, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_1(s, \cdot) \|_{L^\infty(\mathcal{O})} + \| \psi_2(s, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_3(s, \cdot) \|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s < \infty \tag{14}$$

在构造 Tempered-拉回吸引子时,需要下面的假设:对 $\forall \sigma > 0$, 有

$$\begin{aligned}\lim_{r\to-\infty} \mathrm{e}^{\sigma r} \int_{-\infty}^{0} \mathrm{e}^{\lambda s}(& \| G(s + r, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_1(s + r, \cdot) \|_{L^\infty(\mathcal{O})} + \\ & \| \psi_2(s + r, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_3(s + r, \cdot) \|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s = 0\end{aligned} \tag{15}$$

2 解的一致性估计

本节将对问题(11)的解在 $H_g(\mathcal{O})$ 空间上进行一致性估计.

引理 1 假设(3)-(6)式和(14)式成立,则存在 $\varepsilon_0 > 0$, 对 $\forall \varepsilon \in (0, \varepsilon_0)$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D_1 = \{D_1(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, 存在与 ε 无关的 $T = T(\tau, \omega, D_1)$, 使得对 $\forall t \geqslant T$, 方程(11)的解满足:

$$\begin{aligned}\| v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}^\varepsilon) \|_{H_g(\mathcal{O})}^2 \leqslant & M\left(1 + \int_{-\infty}^{0} \mathrm{e}^{\lambda s} | z(\theta_s\omega) |^p \mathrm{d}s\right) + \\ & M\mathrm{e}^{-\lambda\tau} \int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\| G(s, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_1(s, \cdot) \|_{L^\infty(\mathcal{O})} + \| \psi_2(s, \cdot) \|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s\end{aligned} \tag{16}$$

$$\begin{aligned}& \int_{\tau-t}^{\tau} \mathrm{e}^{\lambda s}(\| v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}^\varepsilon) \|_{H_\varepsilon^1(\mathcal{O})}^2 + \| v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}^\varepsilon) + h_\varepsilon z(\theta_{s-\tau}\omega) \|_{L^p(\mathcal{O})}^p)\mathrm{d}s \leqslant \\ & M\mathrm{e}^{\lambda\tau}\left(1 + \int_{-\infty}^{0} \mathrm{e}^{\lambda s} | z(\theta_s\omega) |^p \mathrm{d}s\right) + M\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\| G(s, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_1(s, \cdot) \|_{L^\infty(\mathcal{O})} + \| \psi_2(s, \cdot) \|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s\end{aligned} \tag{17}$$

其中, $v_{\tau-t}^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$; M 依赖 λ, 但与 $\tau, \omega, \varepsilon, D_1$ 无关.

证 将(11)式与 v^ε 在空间 $H_g(\mathcal{O})$ 做内积,可得到

$$\begin{aligned}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| v^\varepsilon \|_{H_g(\mathcal{O})}^2 + (A_\varepsilon v^\varepsilon, v^\varepsilon)_{H_g(\mathcal{O})} + \lambda(v^\varepsilon, v^\varepsilon)_{H_g(\mathcal{O})} = \\ & (f_\varepsilon(t, y, v^\varepsilon + h_\varepsilon z(\theta_t\omega)), v^\varepsilon)_{H_g(\mathcal{O})} + (G_\varepsilon(t, y), v^\varepsilon)_{H_g(\mathcal{O})} - z(\theta_t\omega)(A_\varepsilon h_\varepsilon, v^\varepsilon)_{H_g(\mathcal{O})}\end{aligned} \tag{18}$$

对(18)式左边第一项估计,可得

$$\begin{aligned}& (f_\varepsilon(t, y, v^\varepsilon + h_\varepsilon z(\theta_t\omega)), v^\varepsilon)_{H_g(\mathcal{O})} = \int_{\mathcal{O}} g f(t, y^*, \varepsilon g(y^*) y_{n+1}, v^\varepsilon + h_\varepsilon z(\theta_t\omega))(v^\varepsilon + h_\varepsilon z(\theta_t\omega))\mathrm{d}y - \\ & z(\theta_t\omega)\int_{\mathcal{O}} g f(t, y^*, \varepsilon g(y^*) y_{n+1}, v^\varepsilon + h_\varepsilon z(\theta_t\omega)) h_\varepsilon(y)\mathrm{d}y \leqslant \\ & -\alpha_1 r_1 \int_{\mathcal{O}} | v^\varepsilon + h_\varepsilon z(\theta_t\omega) |^p \mathrm{d}y + r_2 \int_{\mathcal{O}} | \psi_1(t, y^*, \varepsilon g(y^*) y_{n+1}) | \mathrm{d}y +\end{aligned}$$

$$\alpha_2 r_2\int_{\mathcal{O}} |v^\varepsilon + h_\varepsilon z(\theta_t\omega)|^{p-1} |h_\varepsilon z(\theta_t\omega)| \mathrm{d}y + r_2\int_{\mathcal{O}} |\psi_2(t, y^*, \varepsilon g(y^*) y_{n+1})| |h_\varepsilon z(\theta_t\omega)| \mathrm{d}y \leqslant$$
$$-\frac{1}{2}\alpha_1 r_1\int_{\mathcal{O}} |v^\varepsilon + h_\varepsilon z(\theta_t\omega)|^p \mathrm{d}y + c_1(1 + \|\psi_1(t, \cdot)\|_{L^\infty(\mathcal{O})} + \|\psi_2(t, \cdot)\|^2_{L^\infty(\mathcal{O})} + |z(\theta_t\omega)|^p) \tag{19}$$

分别对(18)式左边第二项和最后一项估计，得：

$$(G_\varepsilon(t, y), v^\varepsilon)_{H_g(\mathcal{O})} \leqslant \frac{1}{4}\lambda\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + \frac{1}{\lambda}\|G_\varepsilon(t, y)\|^2_{H_g(\mathcal{O})} \leqslant$$
$$\frac{1}{4}\lambda\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + c_2\|G(t, \cdot)\|^2_{L^\infty(\mathcal{O})} \tag{20}$$

$$-z(\theta_t\omega)(A_\varepsilon h_\varepsilon, v^\varepsilon)_{H_g(\mathcal{O})} \leqslant |z(\theta_t\omega)| |a_\varepsilon(h_\varepsilon, v^\varepsilon)| \leqslant \frac{1}{2}a_\varepsilon(v^\varepsilon, v^\varepsilon) + c_3(1 + |z(\theta_t\omega)|^P) \tag{21}$$

由(18)—(21)式可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + a_\varepsilon(v^\varepsilon, v^\varepsilon) + \frac{3}{2}\lambda\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + \alpha_1 r_1\int_{\mathcal{O}} |v^\varepsilon + h_\varepsilon z(\theta_t\omega)|^p \mathrm{d}y \leqslant$$
$$c_4(1 + \|G(t, \cdot)\|^2_{L^\infty(\mathcal{O})} + \|\psi_1(t, \cdot)\|_{L^\infty(\mathcal{O})} + \|\psi_2(t, \cdot)\|^2_{L^\infty(\mathcal{O})} + |z(\theta_t\omega)|^p) \tag{22}$$

在(22)式两边乘以 $\mathrm{e}^{\lambda t}$，在$(\tau - t, \tau)$上对 t 积分，其中 $t > 0$，再将 ω 替换为$\theta_{-\tau}\omega$，得

$$\|v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} + \int_{\tau-t}^{\tau} \mathrm{e}^{\lambda(s-\tau)} a_\varepsilon(v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}))\mathrm{d}s +$$
$$\frac{1}{2}\lambda\int_{\tau-t}^{\tau} \mathrm{e}^{\lambda(s-\tau)} \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})}\mathrm{d}s +$$
$$\alpha_1 r_1\int_{\tau-t}^{\tau} \mathrm{e}^{\lambda(s-\tau)} \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}) + h_\varepsilon z(\theta_{s-\tau}\omega)\|^p_{L^p(\mathcal{O})}\mathrm{d}s \leqslant$$
$$\mathrm{e}^{-\lambda t}\|v^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} + c_5\mathrm{e}^{-\lambda\tau}\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\|G(s, \cdot)\|^2_{L^\infty(\mathcal{O})} + \|\psi_1(s, \cdot)\|_{L^\infty(\mathcal{O})} + \|\psi_2(s, \cdot)\|^2_{L^\infty(\mathcal{O})})\mathrm{d}s +$$
$$c_5\int_{-\infty}^{0} \mathrm{e}^{\lambda s} |z(\theta_s\omega)|^p \mathrm{d}s + c_5 \tag{23}$$

注意到 $v^\varepsilon_{\tau-t} \in D_1(\tau - t, \theta_{-t}\omega)$，$D_1$ 是 Tempered，则存在 $T = T(\tau, \omega, D_1)$，使得对 $\forall t \geqslant T$，有 $\mathrm{e}^{-\lambda t}\|v^\varepsilon_{\tau-t}\|^2 \leqslant 1$. 结合(23)式和 $\|\cdot\|_{H^1_\varepsilon(\mathcal{O})}$，引理 1 得证.

引理 2　假设(3)—(6)式和(14)式成立，则存在 $\varepsilon_0 > 0$，对 $\forall \varepsilon \in (0, \varepsilon_0)$，$\tau \in \mathbb{R}$，$\omega \in \Omega$，$D_1 = \{D_1(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathscr{D}_1$，存在 $T = T(\tau, \omega, D_1)$，使得对 $\forall t \geqslant T$，方程(11)的解满足

$$\int_{\tau-1}^{\tau}(\|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} + \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}) + h_\varepsilon z(\theta_{s-\tau}\omega)\|^p_{L^p(\mathcal{O})})\mathrm{d}s \leqslant$$
$$M\left(1 + \int_{-\infty}^{0} \mathrm{e}^{\lambda s} |z(\theta_s\omega)|^p \mathrm{d}s\right) +$$
$$M\mathrm{e}^{-\lambda\tau}\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\|G(s, \cdot)\|^2_{L^\infty(\mathcal{O})} + \|\psi_1(s, \cdot)\|_{L^\infty(\mathcal{O})} + \|\psi_2(s, \cdot)\|^2_{L^\infty(\mathcal{O})})\mathrm{d}s \tag{24}$$

证　由引理 1，存在 $T = T(\tau, \omega, D) \geqslant 1$，对 $\forall t \geqslant T$，有

$$\mathrm{e}^{\lambda(\tau-1)}\int_{\tau-1}^{\tau}(\|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} + \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}) + h_\varepsilon z(\theta_{s-\tau}\omega)\|^p_{L^p(\mathcal{O})})\mathrm{d}s \leqslant$$
$$\int_{\tau-t}^{\tau} \mathrm{e}^{\lambda s}(\|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} + \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}) + h_\varepsilon z(\theta_{s-\tau}\omega)\|^p_{L^p(\mathcal{O})})\mathrm{d}s \leqslant$$
$$M\mathrm{e}^{\lambda\tau}\left(1 + \int_{-\infty}^{0} \mathrm{e}^{\lambda s} |z(\theta_s\omega)|^p \mathrm{d}s\right) +$$
$$M\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\|G(s, \cdot)\|^2_{L^\infty(\mathcal{O})} + \|\psi_1(s, \cdot)\|_{L^\infty(\mathcal{O})} + \|\psi_2(s, \cdot)\|^2_{L^\infty(\mathcal{O})})\mathrm{d}s$$

则可以得到(24)式.

引理 3　假设(3)—(6)式和(14)式成立，则存在 $\varepsilon_0 > 0$，对 $\forall \varepsilon \in (0, \varepsilon_0)$，$\tau \in \mathbb{R}$，$\omega \in \Omega$，$D_1 =$

$\{D_1(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathscr{D}_1$，存在 $T=T(\tau, \omega, D_1)$，使得对 $\forall t \geqslant T$，方程(11)的解满足

$$\| v^{\varepsilon}(\tau, \tau-t, \theta_{-\tau}, v_{\tau-t}^{\varepsilon}) \|_{H_{\varepsilon}^1(\mathscr{O})}^2 \leqslant$$

$$M\left(1+\int_{-\infty}^0 \mathrm{e}^{\lambda s} |z(\theta_s \omega)|^p \mathrm{d}s\right)+$$

$$M\mathrm{e}^{-\lambda\tau}\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s}(\| G(s, \cdot) \|_{L^{\infty}(\mathscr{O})}^2 + \| \psi_1(s, \cdot) \|_{L^{\infty}(\mathscr{O})} + \| \psi_2(s, \cdot) \|_{L^{\infty}(\mathscr{O})}^2 + \| \psi_3(s, \cdot) \|_{L^{\infty}(\mathscr{O})}^2)\mathrm{d}s \tag{25}$$

其中，$v_{\tau-t}^{\varepsilon} \in D_1(\tau-t, \theta_{-t}\omega)$；$M$ 仅依赖 λ，与 $\tau, \omega, \varepsilon, D_1$ 无关.

证 让(11)式与 $A_{\varepsilon} v^{\varepsilon}$ 在空间 $H_g(\mathscr{O})$ 上做内积，可得到

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a_{\varepsilon}(v^{\varepsilon}, v^{\varepsilon}) + \| A_{\varepsilon} v^{\varepsilon} \|_{H_g(\mathscr{O})} + \lambda a_{\varepsilon}(v^{\varepsilon}, v^{\varepsilon}) =$$

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} + (G_{\varepsilon}(t, y), A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} - z(\theta_t \omega)(A_{\varepsilon} h_{\varepsilon}, A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} \tag{26}$$

首先，对(26)式中的非线性项进行估计，得

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} =$$

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon}(v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)))_{H_g(\mathscr{O})} -$$

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon} h_{\varepsilon} z(\theta_t \omega))_{H_g(\mathscr{O})} \tag{27}$$

为了对(27)式的右边第一项估计，先进行如下计算：

$$a_{\varepsilon}(h_{\varepsilon} z(\theta_t \omega), h_{\varepsilon} z(\theta_t \omega)) = z^2(\theta_t \omega)\int_{\mathscr{O}} (\boldsymbol{J}^{\mathrm{T}} \nabla h_{\varepsilon})^2 g \,\mathrm{d}y \leqslant c_2 z^2(\theta_t \omega) \tag{28}$$

对于(27)式右边第一项，由文献[1]中的引理 3，并结合(28)式得

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon}(v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)))_{H_g(\mathscr{O})} \leqslant$$

$$c_3(a_{\varepsilon}(v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega), v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)) + \| \psi_3(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2) \leqslant$$

$$c_4(a_{\varepsilon}(v^{\varepsilon}, v^{\varepsilon}) + z^2(\theta_t \omega) + \| \psi_3(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2) \leqslant$$

$$C_4(1+a_{\varepsilon}(v^{\varepsilon}, v^{\varepsilon}) + |z(\theta_t \omega)|^p + \| \psi_3(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2) \tag{29}$$

对于(27)式右边最后一项进行估计，得

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon} h_{\varepsilon} z(\theta_t \omega))_{H_g(\mathscr{O})} \leqslant$$

$$\alpha_2 r_2 \int_{\mathscr{O}} |v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)|^{p-1} |A_{\varepsilon} h_{\varepsilon} z(\theta_t \omega)| \mathrm{d}y + r_2 \int_{\mathscr{O}} |\psi_2(t, y)| |A_{\varepsilon} h_{\varepsilon} z(\theta_t \omega)| \mathrm{d}y \leqslant$$

$$\frac{\alpha_2 r_2 (p-1)}{p}\int_{\mathscr{O}} |v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)|^p \mathrm{d}y + \frac{r_2}{p}|z(\theta_t \omega)|^p \int_{\mathscr{O}} \| h(\cdot) \|_{w^{2,+\infty}(\mathscr{O})}^p \mathrm{d}y +$$

$$\frac{r_2}{2}\int_{\mathscr{O}} \| \psi_2(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2 \mathrm{d}y + |z(\theta_t \omega)|^2 \frac{r_2}{2}\int_{\mathscr{O}} \| h(\cdot) \|_{w^{2,+\infty}(\mathscr{O})}^2 \mathrm{d}y \leqslant$$

$$c_5 \| v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega) \|_{L^p(\mathscr{O})}^p + c_5 |z(\theta_t \omega)|^p + c_5 \| \psi_2(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2 + c_5 \tag{30}$$

由(26)—(30)式可得

$$(f_{\varepsilon}(t, y, v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega)), A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} \leqslant c_6(1+\| v^{\varepsilon}+h_{\varepsilon} z(\theta_t \omega) \|_{L^p(\mathscr{O})}^p + a_{\varepsilon}(v^{\varepsilon}, v^{\varepsilon}) +$$

$$|z(\theta_t \omega)|^p + \| \psi_2(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2 + \| \psi_3(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2) \tag{31}$$

对(26)式中的第二项进行估计，得

$$(G_{\varepsilon}(t, y), A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} \leqslant \frac{1}{4}\| A_{\varepsilon} v^{\varepsilon} \|_{H_g(\mathscr{O})}^2 + \| G_{\varepsilon}(t, y) \|_{H_g(\mathscr{O})}^2 \leqslant$$

$$\frac{1}{4}\| A_{\varepsilon} v^{\varepsilon} \|_{H_g(\mathscr{O})}^2 + c_7 \| G(t, \cdot) \|_{L^{\infty}(\mathscr{O})}^2 \tag{32}$$

对(26)式中的最后一项进行估计，得

$$-z(\theta_t \omega)(A_{\varepsilon} h_{\varepsilon}, A_{\varepsilon} v^{\varepsilon})_{H_g(\mathscr{O})} \leqslant \frac{1}{4}\| A_{\varepsilon} v^{\varepsilon} \|_{H_g(\mathscr{O})}^2 + |z(\theta_t \omega)|^2 \| A_{\varepsilon} h_{\varepsilon} \|_{H_g(\mathscr{O})}^2 \leqslant$$

$$\frac{1}{4}\|A_\varepsilon v^\varepsilon\|_{H_g(\mathcal{O})}^2+c_8\,|z(\theta_t\omega)|^2\|h(\cdot)\|_{W^{2,+\infty}(\mathcal{O})}^2\leqslant$$

$$\frac{1}{4}\|A_\varepsilon v^\varepsilon\|_{H_g(\mathcal{O})}^2+C_8(1+|z(\theta_t\omega)|^p)\tag{33}$$

由(26)－(33)式得

$$\frac{\mathrm{d}}{\mathrm{d}t}a_\varepsilon(v^\varepsilon,v^\varepsilon)+\|A_\varepsilon v^\varepsilon\|_{H_g(\mathcal{O})}+2\lambda a_\varepsilon(v^\varepsilon,v^\varepsilon)_{H_g(\mathcal{O})}\leqslant$$

$$c_9(1+\|v^\varepsilon+h_\varepsilon z(\theta_t\omega)\|_{L^p(\mathcal{O})}^p+a_\varepsilon(v^\varepsilon,v^\varepsilon)+$$

$$\|\psi_2(t,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(t,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(t,\cdot)\|_{L^\infty(\mathcal{O})}^2+|z(\theta_t\omega)|^p)\tag{34}$$

由此易知

$$\frac{\mathrm{d}}{\mathrm{d}t}a_\varepsilon(v^\varepsilon,v^\varepsilon)\leqslant c_9(1+\|v^\varepsilon+h_\varepsilon z(\theta_t\omega)\|_{L^p(\mathcal{O})}^p+a_\varepsilon(v^\varepsilon,v^\varepsilon)+$$

$$\|\psi_2(t,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(t,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(t,\cdot)\|_{L^\infty(\mathcal{O})}^2+|z(\theta_t\omega)|^p)\tag{35}$$

任给定 $t\geqslant 0$，$\tau\in\mathbb{R}$，$\omega\in\Omega$，$s\in(\tau-1,\tau)$，对(35)式在区间(s,τ)上积分，得

$$\begin{aligned}&a_\varepsilon(v^\varepsilon(\tau,\tau-t,\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(\tau,\tau-t,\omega,v_{\tau-t}^\varepsilon))\leqslant\\&a_\varepsilon(v^\varepsilon(s,\tau-t,\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(s,\tau-t,\omega,v_{\tau-t}^\varepsilon))+\\&c_{10}\int_s^\tau(\|v^\varepsilon(\xi,\tau-t,\omega,v_{\tau-t}^\varepsilon)+h_\varepsilon z(\theta_\xi\omega)\|_{L^p(\mathcal{O})}^p+\\&a_\varepsilon(v^\varepsilon(\xi,\tau-t,\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(\xi,\tau-t,\omega,v_{\tau-t}^\varepsilon)))\mathrm{d}\xi+\\&c_{10}\int_s^\tau(1+\|\psi_2(\xi,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(\xi,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(\xi,\cdot)\|_{L^\infty(\mathcal{O})}^2+|z(\theta_\xi\omega)|^p)\mathrm{d}\xi\end{aligned}$$

对上式在区间$(\tau-1,\tau)$上积分，并将 ω 替换成$\theta_{-\tau}\omega$，得到

$$\begin{aligned}&a_\varepsilon(v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon))\leqslant\\&\int_{\tau-1}^\tau a_\varepsilon(v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon))\mathrm{d}s+\\&c_{10}\int_{\tau-1}^\tau(\|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon)+h_\varepsilon z(\theta_{s-\tau}\omega)\|_{L^p(\mathcal{O})}^p+\\&a_\varepsilon(v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon)))\mathrm{d}s+\\&c_{10}\int_{\tau-1}^\tau(1+\|\psi_2(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+|z(\theta_{s-\tau}\omega)|^p)\mathrm{d}s\end{aligned}$$

令 $T=(\tau,\omega,D)\geqslant 1$ 为引理 2 中的正常数，那么由引理 2 可得

$$\begin{aligned}&a_\varepsilon(v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon),v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^\varepsilon))\leqslant\\&c_{11}\mathrm{e}^{-\lambda\tau}\int_{-\infty}^\tau\mathrm{e}^{\lambda s}(\|G(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_1(s,\cdot)\|_{L^\infty(\mathcal{O})}+\|\psi_2(s,\cdot)\|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s+c_{11}\int_{-\infty}^0\mathrm{e}^{\lambda s}\,|z(\theta_s\omega)|^p\mathrm{d}s+\\&c_{11}\int_{\tau-1}^\tau(\|\psi_2(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+|z(\theta_{s-\tau}\omega)|^p)\mathrm{d}s+c_{11}\end{aligned}\tag{36}$$

对于(36)式右边最后一个式子，有

$$\int_{\tau-1}^\tau|z(\theta_{s-\tau}\omega)|^p\mathrm{d}s=\int_{-1}^0|z(\theta_s\omega)|^p\mathrm{d}s\leqslant\mathrm{e}^\lambda\int_{-\infty}^0\mathrm{e}^{\lambda s}\,|z(\theta_s\omega)|^p\mathrm{d}s$$

类似地，有

$$\begin{aligned}&\int_{\tau-1}^\tau(\|\psi_2(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(s,\cdot)\|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s\leqslant\\&\mathrm{e}^\lambda\mathrm{e}^{-\lambda\tau}\int_{\tau-1}^\tau\mathrm{e}^{\lambda s}(\|\psi_2(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(s,\cdot)\|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s\leqslant\\&\mathrm{e}^\lambda\mathrm{e}^{-\lambda\tau}\int_{-\infty}^\tau\mathrm{e}^{\lambda s}(\|\psi_2(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|\psi_3(s,\cdot)\|_{L^\infty(\mathcal{O})}^2+\|G(s,\cdot)\|_{L^\infty(\mathcal{O})}^2)\mathrm{d}s\end{aligned}$$

因此(36)式可化为

$$a_{\varepsilon}(v^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^{\varepsilon}),v^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^{\varepsilon}))\leqslant$$
$$M\left(1+\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\mid z(\theta_s\omega)\mid^{p}\mathrm{d}s\right)+$$
$$M\mathrm{e}^{-\lambda\tau}\int_{-\infty}^{\tau}\mathrm{e}^{\lambda s}(\|\psi_2(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2}+\|\psi_3(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2}+\|G(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2})\mathrm{d}s$$

结合引理 1 可证得(25)式.

3 拉回吸引子的存在性

本节将讨论随机问题(9)和(10)的拉回吸引子的存在性、唯一性.

引理 4 假设(3)-(6)式,(14)式和(15)式成立,则存在 $\varepsilon_0>0$,对 $\forall\varepsilon\in(0,\varepsilon_0)$,随机问题(9)的连续协循环具有一个闭的、可测的 $\mathscr{D}_1$ 拉回吸收集 $K\in\mathscr{D}_1$,对 $\forall\omega\in\mathscr{D}_1$,$\forall\tau\in\mathbb{R}$,$K$ 定义为

$$K(\tau,\omega)=\{u\in L^2(\mathcal{O}):\|u\|_{L^2(\mathcal{O})}^{2}\leqslant L(\tau,\omega)\}$$

其中

$$L(\tau,\omega)=M(1+z^2(\omega))+M\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\mid z(\theta_s\omega)\mid^{p}\mathrm{d}s+$$
$$M\mathrm{e}^{-\lambda\tau}\int_{-\infty}^{\tau}\mathrm{e}^{\lambda s}(\|G(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2}+\|\psi_1(s,\cdot)\|_{L^{\infty}(\mathcal{O})}+$$
$$\|\psi_2(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2}+\|\psi_3(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2})\mathrm{d}s \tag{37}$$

证 任给定 $D_1=\{D_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathscr{D}_1$,定义新的集族 $\hat{D}_1$ 为

$$\hat{D}_1=\{\hat{D}_1(\tau,\omega):\hat{D}_1(\tau,\omega)=\{v\in L^2(\mathcal{O}):\|v\|_{L^2(\mathcal{O})}^{2}\leqslant 2\mid z(\omega)\mid^2\|h_{\varepsilon}\|^2+2\|D(\tau,\omega)\|^2\}\}$$

因为 $z(\omega)$ 是 Tempered,$D_1\in\mathscr{D}_1$,易证得 $\hat{D}_1\in\mathscr{D}_1$. 另外,如果 $u_{\tau-t}^{\varepsilon}\in D_1(\tau-t,\theta_{-t}\omega)$,那么

$$v_{\tau-t}^{\varepsilon}=u_{\tau-t}^{\varepsilon}-h_{\varepsilon}z(\theta_{-t}\omega)\in\hat{D}_1(\tau-t,\theta_{-t}\omega)$$

由引理 3,存在 $T=T(\tau,\omega,D_1)$,使得 $\forall t\geqslant T$,有

$$\|v^{\varepsilon}(\tau,\tau-t,\theta_{-t}\omega,v_{\tau-t}^{\varepsilon})\|_{H_{\varepsilon}^{1}(\mathcal{O})}^{2}\leqslant$$
$$M\left(1+\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\mid z(\theta_t\omega)\mid^{p}\mathrm{d}s\right)+$$
$$M\mathrm{e}^{-\lambda\tau}\int_{-\infty}^{\tau}\mathrm{e}^{\lambda s}(\|G(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2}+\|\psi_1(s,\cdot)\|_{L^{\infty}(\mathcal{O})}+\|\psi_2(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2}+\|\psi_3(s,\cdot)\|_{L^{\infty}(\mathcal{O})}^{2})\mathrm{d}s \tag{38}$$

注意到

$$u^{\varepsilon}(t+\tau,\tau,\theta_{-\tau}\omega,u_{\tau}^{\varepsilon})=v^{\varepsilon}(t+\tau,\tau,\theta_{-\tau}\omega,v_{\tau}^{\varepsilon})+h_{\varepsilon}z(\theta_t\omega)$$

由此可知

$$u^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}^{\varepsilon})=v^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^{\varepsilon})+h_{\varepsilon}z(\omega)$$

结合(37)式可得,对 $u_{\tau-t}^{\varepsilon}\in D_1(\tau-t,\theta_{-t}\omega)$,有

$$\|u^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}^{\varepsilon})\|_{H_{\varepsilon}^{1}(\mathcal{O})}^{2}\leqslant L(\tau,\omega)$$

因此,对 $\forall\tau\in\mathbb{R}$,$\omega\in\Omega$,$D_1\in\mathscr{D}_1$,存在与 ε 无关的 $T=T(\tau,\omega,D_1)\geqslant 1$,使得 $\forall t\geqslant T$,有

$$\Phi_{\varepsilon}(t,\tau-t,\theta_{-t}\omega,D_1(\tau-t,\theta_{-t}\omega))\subset K(\tau,\omega) \tag{39}$$

接下来,证明 $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ 是 Tempered. 任给定 $\delta>0$,$\tau\in\mathbb{R}$,$\omega\in\Omega$,有

$$\lim_{t\to-\infty}\mathrm{e}^{\delta t}\|K(\tau+t,\theta_t\omega)\|_{L^2(\mathcal{O})}^{2}\leqslant\lim_{t\to-\infty}\mathrm{e}^{\delta t}L(\tau+t,\theta_t\omega)$$

结合(14),(15)和(37)式,可知

$$\lim_{t\to-\infty}\mathrm{e}^{\delta t}\|K(\tau+t,\theta_t\omega)\|_{L^2(\mathcal{O})}=0$$

因此 $K(\tau,\omega)$ 在空间 $L^2(\mathcal{O})$ 上是 Tempered. 另外,对 $\forall\tau\in\mathbb{R}$,$L(\tau,\cdot):\Omega\longrightarrow\mathbb{R}$ 是$(\mathscr{F},\mathscr{B})$-可测的. 所

以 K 是 Φ_ε 在 $\mathscr{D}_1$ 中的闭的可测的 $\mathscr{D}_1$ -拉回吸收集.

定理 1　假设(3)－(6)式,(14)式和(15)式成立. 那么存在 $\varepsilon_0 > 0$，对 $\forall \varepsilon \in (0, \varepsilon_0)$，协循环 Φ_ε 在 $L^2(\mathscr{O})$ 有唯一的 $\mathscr{D}_1$ -拉回吸引子 $A_\varepsilon = \{A_\varepsilon(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathscr{D}_1$.

证　首先，由引理4知，Φ_ε 有一闭的、可测的 $\mathscr{D}_1$ -拉回吸收集 K，由(39)式和紧嵌入 $H^1(\mathscr{O}) \circlearrowleft L^2(\mathscr{O})$，根据索伯列夫紧嵌入定理可知，$\Phi_\varepsilon$ 在 $L^2(\mathscr{O})$ 上有一个紧的吸收集. 因此，由文献[1]中吸引子的存在性结论，可得协循环 Φ_ε 存在 $\mathscr{D}_1$ -拉回吸引子.

定理 2　假设(3)－(6)式,(14)式和(15)式成立，协循环 Φ_0 在 $L^2(\mathscr{Q})$ 上有唯一的 $\mathscr{D}_0$ -拉回吸引子

$$A_0 = \{A_0(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathscr{D}_0$$

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Existence of Attractors for Non-Autonomous Stochastic Reaction-Diffusion Equations on Thin Domains

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Abstract: In this paper, we investigate the limiting behavior of non-autonomous stochastic reaction-diffusion equations driven by additive noise defined on thin domains. We prove the existence and uniqueness of the pullback random attractor for the equations in an $n+1$-dimensional thin domain.

Key words: thin domain; stochastic reaction-diffusion equation; additive noise; pullback attractor

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