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广义 HEISENBERG-GREINER p-退化椭圆算子的一类带有余项的含权 Hardy 不等式

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摘要：研究了广义 Heisenberg-Greiner p-退化椭圆算子的一类带有余项的含权 Hardy 不等式的推广问题. 利用散度定理并选择恰当的向量场，得到一类带有余项的含权 Hardy 不等式. 结合逼近的方法，给出了最佳常数的证明，进一步推广了已有的结果.

关 键 词：广义 Heisenberg-Greiner p-退化椭圆算子；带有余项的含权 Hardy 不等式；最佳常数

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A Class of Weighted Hardy Inequalities with Remainder Terms for Generalized p-degenerate Sub-elliptic Heisenberg-Greiner Operators

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Abstract: In this paper, we present the improved versions of weighted Hardy inequalities with remainder terms for the generalized p-degenerate sub-elliptic Heisenberg-Greiner operators. By employing divergence theorem and choosing suitable vector fields, we obtained some weighted Hardy inequalities with remainder terms. Furthermore, the proof ofthe best constants is given by combining with the approximation method, which extend the existing results.

Key words: generalized p-degenerate sub-elliptic Heisenberg-Greiner operators; weighted Hardy inequalities with remainder terms; best constants

近年来，不仅出现了更多关于含权 Hardy 不等式的研究成果[1-5]，而且在齐次群上获得了改进后的 Hardy 不等式[6-7]. 但是关于带余项的 Hardy 不等式的研究文献并不多见. 文献[8]在 Heisenberg 群上建立

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了带有余项的 Hardy 不等式：若 $\Omega \subset \mathbb{H}^n$，$0 \in \Omega$，$p \neq Q$，则对于 $u \in D_0^{1,p}(\Omega \backslash \{0\})$，$R \geqslant R_0$ 存在 $M_0 > 0$，使得 $\sup\limits_{x \in \Omega} d(x) \mathrm{e}^{\frac{1}{M_0}} = R_0 < \infty$，有

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^p \mathrm{d}x \geqslant \left|\frac{Q-p}{p}\right|^p \int_\Omega \psi_p \frac{|u|^p}{d^p} \mathrm{d}x + \frac{p-1}{2p}\left|\frac{Q-p}{p}\right|^{p-2} \int_\Omega \psi_p \frac{|u|^p}{d^p}\left(\ln\left(\frac{R}{d}\right)\right)^{-2} \mathrm{d}x \tag{1}$$

而且当 $2 \leqslant p < Q$ 时，可以得到

$$\sup_{x \in \Omega} d(x) = R_0$$

其中(1)式中的常数 $\left|\frac{Q-p}{p}\right|^p$，$\frac{p-1}{2p}\left|\frac{Q-p}{p}\right|^{p-2}$ 是最佳的.

本文使用类似于文献[8-9]的方法，针对广义 Heisenberg-Greiner p-退化椭圆算子，利用散度定理引入一类性质恰当的向量场，结合逼近的思想，推广了(1)式，得到了广义 Heisenberg-Greiner p-退化椭圆算子的一类带有余项的含权 Hardy 不等式，进一步给出了最佳常数的证明. 这个结果包含了已有的相关结论.

1 预备知识

广义 Heisenberg-Greiner p-退化椭圆算子作为一类具有高奇性的平方和退化椭圆算子[10]，被越来越多的学者所关注，并得到了许多重要的成果[11]. 其构成向量场(见下文)$X_j, Y_j (j=1,2,\cdots,n)$ 在 $k>1$ 时不满足 Hörmander 有限秩条件，从而它的亚椭圆性无法由此导出，增加了研究的难度[12-13]. 以下给出广义 Heisenberg-Greiner p-退化椭圆算子的基本知识.

广义 Heisenberg-Greiner p-退化椭圆算子形为

$$\mathcal{L}_p u = \mathrm{div}_L(|\nabla_L u|^{p-2} \nabla_L u) \tag{2}$$

其中：$\nabla_L = (X_1, \cdots, X_n, Y_1, \cdots, Y_n)$，$\mathrm{div}_L(u_1, \cdots, u_{2n}) = \sum\limits_{j=1}^n (X_j u_j + Y_j u_{n+j})$，$p>1$，这里 $X_j = \frac{\partial}{\partial x_j} + 2k y_j |z|^{2k-2} \frac{\partial}{\partial t}$，$Y_j = \frac{\partial}{\partial y_j} - 2k x_j |z|^{2k-2} \frac{\partial}{\partial t}$，$z_j = x_j + \sqrt{-1} y_j \in \mathbb{C}$，$j=1,2,\cdots,n$，$t \in \mathbb{R}$，$k \geqslant 1$. 注意到，当 $p=2$，$k=1$ 时，$\mathcal{L}_p$ 就成为 Heisenberg 群 $\mathbb{H}^n$ 上的 Kohn Laplacian 算子 $\Delta_{\mathbb{H}^n}$[14]. 当 $p=2$，$k=2,3,\cdots$ 时，$\mathcal{L}_p$ 就成为 Greiner 算子[15]

$$\mathcal{L} = \sum_{j=1}^n (X_j^2 + Y_j^2)$$

设 $\xi = (z, t) = (x, y, t) \in \mathbb{R}^{2n+1}$，相应于(2)式中 $\mathcal{L}_p$ 的一个自然伸缩为

$$\delta_\tau(z, t) = (\tau z, \tau^{2k} t) \qquad \tau > 0 \tag{3}$$

与伸缩(3)式相应的齐次维数是对应的齐次维数 $Q = 2n + 2k$. 由(3)式诱导的一个拟距离为

$$d(z, t) = (|z|^{4k} + t^2)^{\frac{1}{4k}} \tag{4}$$

通过(4)式直接计算知道

$$\nabla_L d = \frac{1}{d^{4k-1}} \begin{pmatrix} |z|^{4k-2} x + t y |z|^{2k-2} \\ |z|^{4k-2} y - t x |z|^{2k-2} \end{pmatrix}$$

$$|\nabla_L d|^p = d^{-(2k-1)p} |z|^{(2k-1)p} = \psi_p, \quad \mathcal{L}_p d = \frac{\psi_p}{d}(Q-1) \tag{5}$$

本文在证明最佳常数时，用到了文献[16]中关于广义 Heisenberg-Greiner p-退化椭圆算子的极坐标变换 $(x, y, t) \longrightarrow (\rho, \theta, \theta_1, \cdots, \theta_{2n-1})$. 若 $u(\xi) = \psi_p v(d(\xi))$，则

$$\int_\Omega u(\xi) \mathrm{d}\xi = s_{n,k} \int_{R_1}^{R_2} \rho^{Q-1} v(\rho) \mathrm{d}\rho \tag{6}$$

其中 $s_{n,k} = \omega_n \int_0^\pi (\sin\theta)^{\frac{Q-2}{2k}} \mathrm{d}\theta$，$\omega_n$ 是在 $\mathbb{R}^{2n}$ 中单位 Euclidean 球的 $2n$-Lebesgue 测度.

另外，定义中心在$\{0\}\subset\mathbb{R}^{2n+1}$，半径为 R 的拟开球为$B_R(\xi)=\{\xi\in\mathbb{R}^{2n+1}\mid d(\xi)<R\}$.

令 $\Omega\subset\mathbb{R}^{2n+1}$，$\Omega$ 是开子集，$C_0^k(\Omega)$ 表示 $C^k(\Omega)$ 中具有紧支集的函数构成的集合，$D_0^{1,p}(\Omega)(1<p<\infty)$ 是 $C_0^\infty(\Omega)$ 在范数

$$\|u\|_{D_0^{1,p}}=\left(\int_\Omega|\nabla_L u|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$

下的完备化.

2　两个重要引理

为证明(23)式中常数的最佳性，在这部分给出两个重要引理. 首先定义测试函数及相关函数.

对于一个任意小的 $\delta>0$，定义测试函数 $\varphi(\xi)\in C_0^\infty(\Omega)$ 满足 $0\leqslant\varphi\leqslant1$，$|\nabla_L\varphi|<2\dfrac{|\nabla_L d|}{d}$ 且

$$\varphi(\xi)=\begin{cases}1 & \xi\in B_{\frac{\delta}{2}}(\xi)\\ 0 & \xi\in\Omega\backslash B_{\frac{\delta}{2}}(\xi)\end{cases}\tag{7}$$

对于一个任意小的 $\varepsilon>0$，定义下列函数

$$\omega_\varepsilon=d^{-A+\varepsilon}\eta^{-\theta}\left(\frac{d}{R}\right),\ \frac{1}{p}<\theta<\frac{2}{p},\ A=\frac{Q-\alpha}{p}$$

$$V_\varepsilon(\xi)=\varphi(\xi)\omega_\varepsilon$$

$$J_\gamma(\varepsilon)=\int_\Omega\varphi^p(\xi)\frac{|\nabla_L d|^\beta}{d^{Q-p\varepsilon}}\eta^{-\gamma}\left(\frac{d}{R}\right)\mathrm{d}\xi,\ \gamma=p\theta-2$$

$$I(V_\varepsilon)=\int_\Omega\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_\varepsilon|^p\mathrm{d}\xi-|A|^p\int_\Omega\frac{|\nabla_L d|^\beta}{d^\alpha}|V_\varepsilon|^p\mathrm{d}\xi$$

其中 $\eta(s)=-\dfrac{1}{\ln s}$，$s\in(0,1)$. 容易知道当$\sup\limits_{\xi\in\Omega}d(\xi)<R$，$\xi\in\Omega$ 时，就会存在常数 $M>0$，使得

$$0\leqslant\eta\left(\frac{d(\xi)}{R}\right)\leqslant M\tag{8}$$

引理 1　对于 $\varepsilon>0$，以下式子成立：

(i) $c\varepsilon^{-1-\gamma}\leqslant J_\gamma(\varepsilon)\leqslant C\varepsilon^{-1-\gamma}$，$\gamma>-1$；

(ii) $J_\gamma(\varepsilon)=\dfrac{p\varepsilon}{\gamma+1}J_{\gamma+1}(\varepsilon)+O_\varepsilon(1)$，$\gamma>-1$；

(iii) $J_\gamma(\varepsilon)=O_\varepsilon(1)$，$\gamma<-1$.

证　设 $\rho=R\tau^{\frac{1}{\varepsilon}}$，有 $\mathrm{d}\rho=\dfrac{1}{\varepsilon}R\tau^{\frac{1}{\varepsilon}-1}\mathrm{d}\tau$，$\eta^{-\gamma}(\tau^{\frac{1}{\varepsilon}})=\varepsilon^{-\gamma}\eta^{-\gamma}(\tau)$，由(6)式得

$$\begin{aligned}J_\gamma(\varepsilon)\leqslant{}& s_{n,k}\int_{\rho<\delta}\rho^{-1+p\varepsilon}\eta^{-\gamma}\left(\frac{\rho}{R}\right)\mathrm{d}\rho=\\ & s_{n,k}\int_0^{\left(\frac{\delta}{R}\right)^\varepsilon}(R\tau^{\frac{1}{\varepsilon}})^{-1+p\varepsilon}\eta^{-\gamma}\left(\frac{R\tau^{\frac{1}{\varepsilon}}}{R}\right)\frac{1}{\varepsilon}R\tau^{\frac{1}{\varepsilon}-1}\mathrm{d}\tau=\\ & s_{n,k}R^{p\varepsilon}\varepsilon^{-1-\gamma}\int_0^{\left(\frac{\delta}{R}\right)^\varepsilon}\tau^{p-1}\eta^{-\gamma}(\tau)\mathrm{d}\tau\end{aligned}\tag{9}$$

容易知道

$$\int_a^b\frac{\eta^{\gamma+1}(s)}{s}\mathrm{d}s=\frac{1}{\gamma}(\eta^\gamma(b)-\eta^\gamma(a))\tag{10}$$

通过(10)式可以知道(9)式中

$$\int_0^{\left(\frac{\delta}{R}\right)^\varepsilon}\tau^{p-1}\eta^{-\gamma}(\tau)\mathrm{d}\tau$$

是有限的，这样(i)右边不等式得到证明. 利用(5)式，由(7)式知道在 $B_{\frac{\delta}{2}}(\xi)$ 上，$\varphi=1$，从而

$$J_{\gamma}(\varepsilon)\geqslant\int_{B_{\frac{\delta}{2}}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-\gamma}\left(\frac{d}{R}\right)\mathrm{d}\xi=$$

$$s_{n,k}R^{p\varepsilon}\varepsilon^{-1-\gamma}\int_0^{\left(\frac{\delta}{2R}\right)^{\varepsilon}}\tau^{p-1}\eta^{-\gamma}(\tau)\mathrm{d}\tau \tag{11}$$

同样在(11)式中，利用(10)式证得(i)左边不等式成立.

容易知道

$$\nabla_L\eta^{\gamma}\left(\frac{d}{R}\right)=\gamma\frac{\eta^{\gamma+1}\left(\frac{d}{R}\right)\nabla_L d}{d},\quad\frac{\mathrm{d}\eta^{\gamma}\left(\frac{\rho}{R}\right)}{\mathrm{d}\rho}=\gamma\frac{\eta^{\gamma+1}\left(\frac{\rho}{R}\right)}{\rho},\ \gamma\in\mathbb{R} \tag{12}$$

设 $\Omega_{\eta}=\{\xi\in\Omega\mid d(\xi)>\eta,\ \eta>0\}$，有

$$-\int_{d=\eta}\left(\frac{\varphi^{p}|\nabla_L d|^{\beta-2}\nabla_L d}{d^{Q-1-p\varepsilon}}\right)\eta^{-\gamma-1}\left(\frac{d}{R}\right)\nabla_L d\cdot\vec{n}\,\mathrm{d}S\to 0,\ \eta\to 0$$

再利用(12)式，得到

$$\int_{\Omega}\mathrm{div}_L\left(\frac{\varphi^{p}|\nabla_L d|^{\beta-2}\nabla_L d}{d^{Q-1-p\varepsilon}}\right)\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi=$$

$$-\int_{\Omega}\frac{\varphi^{p}|\nabla_L d|^{\beta-2}}{d^{Q-1-p\varepsilon}}\left\langle\nabla_L d,\ \nabla_L\eta^{-\gamma-1}\left(\frac{d}{R}\right)\right\rangle\mathrm{d}\xi=$$

$$(\gamma+1)\int_{\Omega}\frac{\varphi^{p}|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-\gamma}\left(\frac{d}{R}\right)\mathrm{d}\xi=$$

$$(\gamma+1)J_{\gamma}(\varepsilon) \tag{13}$$

又由于

$$\int_{\Omega}\mathrm{div}_L\left(\frac{\varphi^{p}|\nabla_L d|^{\beta-2}\nabla_L d}{d^{Q-1-p\varepsilon}}\right)\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi=$$

$$p\int_{\Omega}\varphi^{p-1}\frac{|\nabla_L d|^{\beta-2}\langle\nabla_L d,\ \nabla_L\varphi\rangle}{d^{Q-1-p\varepsilon}}\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi+$$

$$(1-Q+p\varepsilon+Q-1)\int_{\Omega}\varphi^{p}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi=$$

$$p\int_{\Omega}\varphi^{p-1}\frac{|\nabla_L d|^{\beta-2}\langle\nabla_L d,\ \nabla_L\varphi\rangle}{d^{Q-1-p\varepsilon}}\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi+p\varepsilon J_{\gamma+1}(\varepsilon) \tag{14}$$

其中通过(6)式与(7)式知道

$$p\int_{\Omega}\varphi^{p-1}\frac{|\nabla_L d|^{\beta-2}\langle\nabla_L d,\ \nabla_L\varphi\rangle}{d^{Q-1-p\varepsilon}}\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi\leqslant$$

$$2p\int_{B_{\frac{\delta}{2}}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi\leqslant$$

$$2ps_{n,k}\int_{B_{\frac{\delta}{2}}(\xi)}\rho^{-Q+p\varepsilon}\eta^{-\gamma-1}\left(\frac{\rho}{R}\right)\rho^{Q-1}\mathrm{d}\rho=$$

$$2ps_{n,k}\int_{B_{\frac{\delta}{2}}(\xi)}\rho^{p\varepsilon-1}\eta^{-\gamma-1}\left(\frac{\rho}{R}\right)\mathrm{d}\rho$$

根据(i)得到

$$\int_{B_{\frac{\delta}{2}}(\xi)}\rho^{p\varepsilon-1}\eta^{-\gamma-1}\left(\frac{\rho}{R}\right)\mathrm{d}\rho=O_{\varepsilon}(1)$$

从而

$$p\int_{\Omega}\varphi^{p-1}\frac{|\nabla_L d|^{\beta-2}\langle\nabla_L d,\nabla_L\varphi\rangle}{d^{Q-1-p\varepsilon}}\eta^{-\gamma-1}\left(\frac{d}{R}\right)\mathrm{d}\xi=O_{\varepsilon}(1)$$

因此，结合(13)式和(14)式有

$$(\gamma+1)J_{\gamma}(\varepsilon)=p\varepsilon J_{\gamma+1}(\varepsilon)+O_{\varepsilon}(1)$$

这样(ii)得到证明.

利用极坐标变换(6)式，有

$$J_{\gamma}(\varepsilon)\leqslant\int_{B_{\frac{\delta}{2}}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-\gamma}\left(\frac{d}{R}\right)\mathrm{d}\xi=$$
$$s_{n,k}\int_0^{\delta}\rho^{-Q+p\varepsilon}\eta^{-\gamma}\left(\frac{\rho}{R}\right)\rho^{Q-1}\mathrm{d}\rho=$$
$$s_{n,k}\int_0^{\delta}\rho^{-1+p\varepsilon}\eta^{-\gamma}\left(\frac{\rho}{R}\right)\mathrm{d}\rho \tag{15}$$

当 $\gamma<-1$ 时，通过(10)式可以知道(15)式是有限的，从而在(15)式两边取 $\varepsilon\to0$，证得(iii)成立.

引理 2　对于 $\varepsilon\to0$，以下式子成立

(i) $I(V_{\varepsilon})\leqslant\dfrac{\theta(p-1)}{2}|A|^{p-2}J_{p\theta-2}(\varepsilon)+O_{\varepsilon}(1)$；

(ii) $\displaystyle\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_{\varepsilon}|^p\mathrm{d}\xi\leqslant|A|^pJ_{p\theta}(\varepsilon)+O_{\varepsilon}(\varepsilon^{1-p\theta})$.

证　已知 $\nabla_L V_{\varepsilon}(\xi)=\varphi(\xi)\nabla_L\omega_{\varepsilon}+\omega_{\varepsilon}\nabla_L\varphi$. 及

$$|a+b|^p\leqslant|a|^p+c_p(|a|^{p-1}|b|+|b|^p),\ a,b\in\mathbb{R}^{2n},\ p>1 \tag{16}$$

利用(16)式，有

$$|\nabla_L V_{\varepsilon}|^p\leqslant\varphi^p|\nabla_L\omega_{\varepsilon}|^p+c_p(|\nabla_L\varphi|\omega_{\varepsilon}\varphi^{p-1}|\nabla_L\omega_{\varepsilon}|^{p-1}+|\nabla_L\varphi|^p|\omega_{\varepsilon}|^p)\leqslant$$
$$\varphi^p|\nabla_L\omega_{\varepsilon}|^p+c_p\left(\frac{|2\nabla_L d|}{d}\omega_{\varepsilon}\varphi^{p-1}|\nabla_L\omega_{\varepsilon}|^{p-1}+\left(\frac{|2\nabla_L d|}{d}\right)^p|\omega_{\varepsilon}|^p\right)$$

利用

$$\nabla_L\omega_{\varepsilon}=-d^{-A+\varepsilon-1}\nabla_L d\eta^{-\theta}\left(\frac{d}{R}\right)\left(A-\varepsilon+\theta\eta\left(\frac{d}{R}\right)\right)$$

得到

$$\int_{\Omega}\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_{\varepsilon}|^p\mathrm{d}\xi\leqslant\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\varphi^p\eta^{-p\theta}\left(\frac{d}{R}\right)\left|A-\left(\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right)\right|^p\mathrm{d}\xi+$$
$$2c_p\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\varphi^{p-1}\eta^{-p\theta}\left(\frac{d}{R}\right)\left|A-\left(\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right)\right|^{p-1}\mathrm{d}\xi+$$
$$2^pc_p\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-p\theta}\left(\frac{d}{R}\right)\mathrm{d}\xi:=\Pi_A+\Pi_{A1}+\Pi_{A2} \tag{17}$$

由于 $\left|A-\left(\varepsilon-\theta\eta\left(\dfrac{d}{R}\right)\right)\right|$ 是有界的，通过(8)式得到

$$\Pi_{A1}\leqslant C\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\varphi^{p-1}\eta^{-p\theta}\left(\frac{d}{R}\right)\left|A-\left(\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right)\right|^{p-1}\mathrm{d}\xi\leqslant$$
$$C\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\varphi^{p-1}\eta^{-p\theta}\left(\frac{d}{R}\right)\mathrm{d}\xi$$
$$\Pi_{A2}\leqslant C\int_{B_{\delta}(\xi)}\frac{|\nabla_L d|^{\beta}}{d^{Q-p\varepsilon}}\eta^{-p\theta}\left(\frac{d}{R}\right)\mathrm{d}\xi$$

利用引理 1 的(ii)知道 $\Pi_{A1},\Pi_{A2}=O_{\varepsilon}(1)$，$\varepsilon\to0$.

结合(17)式有

$$I(V_\varepsilon)=\int_{B_\delta(\xi)}\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_\varepsilon|^p\mathrm{d}\xi-|A|^p J_{p\theta}(\varepsilon)\leqslant$$

$$\Pi_A-|A|^p J_{p\theta}(\varepsilon)+O_\varepsilon(1)=\Pi_B+O_\varepsilon(1) \tag{18}$$

其中

$$\Pi_B=\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^{Q-p\varepsilon}}\varphi^p\eta^{-p\theta}\left(\frac{d}{R}\right)\left(\left|A-\left(\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right)\right|^p-|A|^p\right)\mathrm{d}\xi$$

设

$$\zeta=\varepsilon-\theta\eta\left(\frac{d}{R}\right)$$

有 $\zeta<A$. 利用 Taylor 公式，得到

$$|A-\zeta|^p-|A|^p\leqslant-pA|A|^{p-2}\zeta+\frac{p(p-1)}{2}|A|^{p-2}\zeta^2+C|\zeta|^3$$

这样

$$\Pi_B\leqslant\Pi_{B1}+\Pi_{B2}+\Pi_{B3} \tag{19}$$

其中

$$\Pi_{B1}=-pA|A|^{p-2}\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^{Q-p\varepsilon}}\varphi^p\eta^{-p\theta}\left(\frac{d}{R}\right)\left(\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right)\mathrm{d}\xi$$

$$\Pi_{B2}=\frac{p(p-1)}{2}|A|^{p-2}\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^{Q-p\varepsilon}}\varphi^p\eta^{-p\theta}\left(\frac{d}{R}\right)\left(\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right)^2\mathrm{d}\xi$$

$$\Pi_{B3}=C\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^{Q-p\varepsilon}}\varphi^p\eta^{-p\theta}\left(\frac{d}{R}\right)\left|\varepsilon-\theta\eta\left(\frac{d}{R}\right)\right|^3\mathrm{d}\xi$$

以下证明

$$\Pi_{B1},\ \Pi_{B3}=O_\varepsilon(1),\ \varepsilon\to0 \tag{20}$$

在引理 1 的 (ⅱ) 中，取 $\gamma=-1+p\theta$ 得到

$$\Pi_{B1}=-pA|A|^{p-2}(\varepsilon J_{p\theta}(\varepsilon)-\theta J_{p\theta-1}(\varepsilon))=$$
$$-pA|A|^{p-2}(\varepsilon J_{p\theta}(\varepsilon)-\varepsilon J_{p\theta}(\varepsilon)+O_\varepsilon(1))=O_\varepsilon(1)$$

利用不等式

$$(a-b)^3\leqslant(|a|+|b|)^3\leqslant c(|a|^3+|b|^3)$$

有

$$\Pi_{B3}\leqslant c\varepsilon^3J_{p\theta}(\varepsilon)+cJ_{p\theta-3}(\varepsilon)\qquad \varepsilon>0$$

由 $1<p\theta<2$，结合引理 1 的 (ⅰ) 及 (ⅲ)，得到 $\Pi_{B3}=O_\varepsilon(1)$. 在引理 1 的 (ⅱ) 中取 $\gamma=p\theta-1>-1$ 后，再次取 $\gamma=p\theta-2>-1$，有

$$\Pi_{B2}=\frac{p(p-1)}{2}|A|^{p-2}\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^{Q-p\varepsilon}}\varphi^p\eta^{-p\theta}\left(\frac{d}{R}\right)\left(\varepsilon^2-2\varepsilon\theta\eta\left(\frac{d}{R}\right)+\theta^2\eta^2\left(\frac{d}{R}\right)\right)\mathrm{d}\xi=$$
$$\frac{p(p-1)}{2}|A|^{p-2}\left(\varepsilon^2J_{p\theta}(\varepsilon)-2\varepsilon\theta J_{p\theta-1}(\varepsilon)+\theta^2\left(\frac{p\theta-1}{p\theta}+\frac{1}{p\theta}\right)J_{p\theta-2}(\varepsilon)\right)=$$
$$\frac{p(p-1)}{2}|A|^{p-2}\left(\varepsilon^2J_{p\theta}(\varepsilon)-\varepsilon\theta\frac{p\varepsilon}{p\theta}J_{p\theta}(\varepsilon)-\varepsilon\theta J_{p\theta-1}(\varepsilon)+\frac{\theta^2(p\theta-1)}{p\theta}\frac{p\varepsilon}{p\theta-1}J_{p\theta-1}(\varepsilon)+\frac{\theta}{p}J_{p\theta-2}(\varepsilon)+O_\varepsilon(1)\right)=$$
$$\frac{\theta(p-1)}{2}|A|^{p-2}J_{p\theta-2}(\varepsilon)+O_\varepsilon(1) \tag{21}$$

结合(18)－(21) 式，得到引理 2 的 (ⅰ). 结合(18),(21) 式及引理 2 的 (ⅰ)，有

$$\int_{B_\delta(\xi)}\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_\varepsilon|^p\mathrm{d}\xi=$$

$$I(V_\varepsilon)+|A|^pJ_{p\theta}(\varepsilon)\leqslant$$

$$
\begin{aligned}
&|A|^p J_{p\theta}(\varepsilon)+\frac{\theta(p-1)}{2}|A|^{p-2}J_{p\theta-2}(\varepsilon)+O_\varepsilon(1)\leqslant\\
&|A|^p J_{p\theta}(\varepsilon)+O_\varepsilon(\varepsilon^{1-p\theta})
\end{aligned}
$$

因此，引理 2 的(ⅱ)成立.

3　一类带有余项的含权 Hardy 不等式

定理 1　若 $1<p<\infty$，$\alpha\neq Q$，$\beta>2-Q$，$a,b\in\mathbb{R}$，则对于 $u\in C_0^\infty(\Omega\backslash\{0\})$，$R\geqslant R_0$，有

$$
\begin{aligned}
&\int_\Omega\left(\frac{|z|}{d}\right)^{(\beta-p)(2k-1)}\left(1+\frac{a}{\left(\ln\left(\frac{R}{d}\right)\right)}+\frac{b}{\left(\ln\left(\frac{R}{d}\right)\right)^2}\right)\frac{|\nabla_L u|^p}{d^{\alpha-p}}\mathrm{d}\xi\geqslant\\
&|A|^p\int_\Omega\left(\frac{|z|}{d}\right)^{\beta(2k-1)}\frac{|u|^p}{d^\alpha}\mathrm{d}\xi+a|A|^p\int_\Omega\left(\frac{|z|}{d}\right)^{\beta(2k-1)}\frac{|u|^p}{d^\alpha\ln\left(\frac{R}{d}\right)}\mathrm{d}\xi+\\
&\left(\frac{p-1}{2p}|A|^{p-2}+aA|A|^{p-2}+b|A|^p\right)\int_\Omega\left(\frac{|z|}{d}\right)^{\beta(2k-1)}\frac{|u|^p}{d^\alpha\ln\left(\frac{R}{d}\right)^2}\mathrm{d}\xi
\end{aligned}
\tag{22}
$$

特别地，在(22)式中取 $a=b=0$，有下列带有余项的权 Hardy 不等式

$$
\begin{aligned}
&\int_\Omega\left(\frac{|z|}{d}\right)^{(\beta-p)(2k-1)}\frac{|\nabla_L u|^p}{d^{\alpha-p}}\mathrm{d}\xi\geqslant\\
&|A|^p\int_\Omega\left(\frac{|z|}{d}\right)^{\beta(2k-1)}\frac{|u|^p}{d^\alpha}\mathrm{d}\xi+\frac{p-1}{2p}|A|^{p-2}\int_\Omega\left(\frac{|z|}{d}\right)^{\beta(2k-1)}\frac{|u|^p}{d^\alpha\ln\left(\frac{R}{d}\right)^2}\mathrm{d}\xi
\end{aligned}
\tag{23}
$$

(23)式中的常数是最佳的，其中 $A=\dfrac{Q-\alpha}{p}$.

证　(22),(23)式的证明见文献[17]中第三部分定理 1 的证明.

以下证明(23)式中常数的最佳性.

1）通过引理 2 的(ⅱ)，得到

$$
\frac{\displaystyle\int_{B_\delta(\xi)}\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_\varepsilon|^p\mathrm{d}\xi}{\displaystyle\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^\alpha}|V_\varepsilon|^p\mathrm{d}\xi}\leqslant\frac{|A|^p J_{p\theta}(\varepsilon)+O_\varepsilon(\varepsilon^{1-p\theta})}{\displaystyle\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^\alpha}\left|\varphi d^{-A+\varepsilon}\eta^{-\theta}\left(\frac{d}{R}\right)\right|^p\mathrm{d}\xi}=\frac{|A|^p(1+c\varepsilon^2)J_{p\theta}(\varepsilon)+O_\varepsilon(1)}{J_{p\theta}(\varepsilon)}
$$

已知当 $\varepsilon\to 0$ 时有 $J_{p\theta}(\varepsilon)\to\infty$，所以当 $\varepsilon\to 0$ 时，有

$$
\frac{\displaystyle\int_{B_\delta(\xi)}\frac{|\nabla_L d|^{\beta-p}}{d^{\alpha-p}}|\nabla_L V_\varepsilon|^p\mathrm{d}\xi}{\displaystyle\int_{B_\delta(\xi)}\frac{|\nabla_L d|^\beta}{d^\alpha}|V_\varepsilon|^p\mathrm{d}\xi}\to|A|^p
$$

2）通过引理 2 的(ⅰ)，得到

$$
\frac{I(V_\varepsilon)}{J_{p\theta-2}(\varepsilon)}\leqslant\frac{\frac{\theta(p-1)}{2}|A|^{p-2}J_{p\theta-2}(\varepsilon)+O_\varepsilon(1)}{J_{p\theta-2}(\varepsilon)}
$$

已知当 $\varepsilon\to 0$ 时，有 $J_{p\theta-2}(\varepsilon)\to\infty$，所以由引理 1 的(ⅰ)，得到当 $\varepsilon\to 0$ 时，$\dfrac{I(V_\varepsilon)}{J_{p\theta-2}(\varepsilon)}\longrightarrow\dfrac{\theta(p-1)}{2}|A|^{p-2}$.

于是当 $\theta\to\dfrac{1}{p}$ 时 $\dfrac{I(V_\varepsilon)}{J_{p\theta-2}(\varepsilon)}\longrightarrow\dfrac{(p-1)}{2p}|A|^{p-2}$.

综合 1),2),(23) 式中常数的最佳性得证.

注 1 在(23) 式中,取 $k=1$,$\alpha=p$,$\beta=p$ 时,得到(1) 式.

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