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一类混合型交换四元数矩阵实表示的性质及应用①

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摘要: 在引入混合型交换四元数及混合型交换四元数矩阵概念的基础上,首先,证明了混合型交换四元数和实数域上的 4 阶矩阵是同构的,将对混合型交换四元数的研究转化为对实数域上 4 阶矩阵的研究.其次,在混合型交换四元数矩阵和实数域上 $4n$ 阶矩阵同构的基础上,将对混合型交换四元数矩阵的研究转化为对实数域上 $4n$ 阶矩阵的研究.利用实矩阵的性质得到混合型交换四元数矩阵实表示的系列性质,并给出了混合型交换四元数矩阵可逆的等价条件.以混合型交换四元数矩阵实表示的性质为基础,得到混合型交换四元数矩阵复特征值的个数及特征值存在的充分必要条件,并将实数域上的盖尔圆盘定理推广到混合型交换四元数矩阵上.最后,利用具体的数值算例验证了混合型交换四元数矩阵盖尔圆盘定理的正确性和有效性.

关 键 词: 混合型交换四元数矩阵;实表示;矩阵特征值;盖尔圆盘定理

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四元数量子力学是现代量子力学的一个重要分支,它是一门建立在非交换四元数代数系统上的量子力学,区别于一般复量子力学.对非交换四元数代数系统的研究已取得丰硕的成果[1-5].随着非交换四元数研究的深入,分裂四元数代数系统逐步引起学者的重视,在分裂四元数系统的研究方面取得了阶段性成果[6-9].然而对乘法满足交换性的交换四元数代数理论的研究成果还不多.文献[10-11]首次研究了交换四元数代数系统,并对交换四元数进行了分类,主要分为椭圆型、抛物型、双曲型和混合型等具体类型.文献[12]研究了椭圆型交换四元数及其矩阵,得到系列结果.本文研究的是一类混合型交换四元数及其矩阵,以混合型交换四元数的实表示及矩阵的计算为基础,推导出混合型交换四元数矩阵实表示的系列定理和数值计算性质.

1　混合型交换四元数及其实表示形式

设

$$\boldsymbol{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

其中 $i^2 = -1$, $j^2 = -1$, $k^2 = 1$, $ijk = 1$, $ij = ji = k$, $jk = kj = -i$, $ki = ik = -j$, $\mathbb{R}$ 表示实数域,称满足条件的交换四元数 q 为混合型交换四元数[10].

定义 1[12]　设 $q = q_0 + iq_1 + jq_2 + kq_3 \in \boldsymbol{H}$,则混合型交换四元数 q 的范数为

$$\| q \| = \sqrt[4]{| qq^{(1)} q^{(2)} q^{(3)} |} = \sqrt[4]{(q_0^2 + q_1^2 - q_2^2 - q_3^2)^2 + (2q_0 q_2 + 2q_1 q_3)^2}$$

定理 1　任一混合型交换四元数均可表示为 $\mathbb{R}$ 上的 4 阶矩阵.

证　设 $q = q_0 + iq_1 + jq_2 + kq_3 \in \boldsymbol{H}$, $q_0, q_1, q_2, q_3 \in \mathbb{R}$. 定义映射 $\varphi_q : \boldsymbol{H} \longrightarrow \boldsymbol{H}$, $\varphi_q(r) = rq$. 任给

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$r \in \boldsymbol{H}$，则 φ_q 为双射，且：

$$\varphi_q(1) = 1q = q_0 + iq_1 + jq_2 + kq_3$$
$$\varphi_q(i) = iq = -q_1 + iq_0 - jq_3 + kq_2$$
$$\varphi_q(j) = jq = -q_2 - iq_3 + jq_0 + kq_1$$
$$\varphi_q(k) = kq = q_3 - iq_2 - jq_1 + kq_0$$

依此映射，可定义混合型交换四元数集合为 4 阶实矩阵集合

$$M_{4\times4}(\mathbb{R}) = \left\{ \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ q_3 & -q_2 & -q_1 & q_0 \end{pmatrix} : q_0, q_1, q_2, q_3 \in \mathbb{R} \right\}$$

的子集合，$\boldsymbol{H}$ 和 $M_{4\times4}(\mathbb{R})$ 本质是相同的，$\mathbb{R}$ 上 4 阶矩阵的性质即为 $\boldsymbol{H}$ 上混合型交换四元数的性质，故对混合型交换四元数的研究可转化为对 $\mathbb{R}$ 上 4 阶矩阵的研究.

称矩阵 $\begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ q_3 & -q_2 & -q_1 & q_0 \end{pmatrix}$ 为混合型交换四元数 q 的实表示形式，记作 $q^{\mathbb{R}}$.

2 混合型交换四元数矩阵实表示的性质及应用

令 $M_{n\times n}(\mathbb{R})$ 表示 $\mathbb{R}$ 上的 n 阶矩阵，$M_{n\times n}(\boldsymbol{H})$ 表示 $\boldsymbol{H}$ 上 n 阶混合型交换四元数矩阵. 设

$$\boldsymbol{A} = \boldsymbol{A}_0 + i\boldsymbol{A}_1 + j\boldsymbol{A}_2 + k\boldsymbol{A}_3 \in M_{n\times n}(\boldsymbol{H}) \qquad \boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3 \in M_{n\times n}(\mathbb{R})$$

定义混合型交换四元数矩阵 $\boldsymbol{A}$ 的实表示为 $\begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix} \in M_{4n\times4n}(\mathbb{R})$，记作 $\boldsymbol{A}^{\mathbb{R}}$，其中

$$M_{4n\times4n}(\mathbb{R}) = \left\{ \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix} \Bigg| : \boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_1, \boldsymbol{A}_1 \in M_{n\times n}(\mathbb{R}) \right\}$$

则 $M_{n\times n}(\boldsymbol{H})$ 和 $M_{4n\times4n}(\mathbb{R})$ 本质上是相同的. $\mathbb{R}$ 上 $4n$ 阶矩阵的性质即为 $\boldsymbol{H}$ 上混合型交换四元数矩阵的性质. 对混合型交换四元数矩阵的研究可转化为对 $\mathbb{R}$ 上 $4n$ 阶矩阵的研究.

性质 1 设 $\boldsymbol{A}, \boldsymbol{B} \in M_{n\times n}(\boldsymbol{H})$，如果 $\boldsymbol{AB} = \boldsymbol{I}_n$，则 $\boldsymbol{BA} = \boldsymbol{I}_n$，其中 $\boldsymbol{I}_n$ 为 n 阶单位矩阵.

证

因为：

$$\boldsymbol{A} = \boldsymbol{A}_0 + i\boldsymbol{A}_1 + j\boldsymbol{A}_2 + k\boldsymbol{A}_3 \qquad \boldsymbol{B} = \boldsymbol{B}_0 + i\boldsymbol{B}_1 + j\boldsymbol{B}_2 + k\boldsymbol{B}_3$$
$$\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{B}_0, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3 \in M_{n\times n}(\mathbb{R}) \qquad \boldsymbol{AB} = \boldsymbol{I}_n$$

则

$$\boldsymbol{AB} = (\boldsymbol{A}_0\boldsymbol{B}_0 - \boldsymbol{A}_1\boldsymbol{B}_1 - \boldsymbol{A}_2\boldsymbol{B}_2 + \boldsymbol{A}_3\boldsymbol{B}_3) + i(\boldsymbol{A}_0\boldsymbol{B}_1 + \boldsymbol{A}_1\boldsymbol{B}_0 - \boldsymbol{A}_2\boldsymbol{B}_3 - \boldsymbol{A}_3\boldsymbol{B}_2) +$$
$$j(\boldsymbol{A}_0\boldsymbol{B}_2 - \boldsymbol{A}_1\boldsymbol{B}_3 + \boldsymbol{A}_2\boldsymbol{B}_0 - \boldsymbol{A}_3\boldsymbol{B}_1) + k(\boldsymbol{A}_0\boldsymbol{B}_3 + \boldsymbol{A}_1\boldsymbol{B}_2 + \boldsymbol{A}_2\boldsymbol{B}_1 + \boldsymbol{A}_3\boldsymbol{B}_0)$$

故：

$$\boldsymbol{A}_0\boldsymbol{B}_0 - \boldsymbol{A}_1\boldsymbol{B}_1 - \boldsymbol{A}_2\boldsymbol{B}_2 + \boldsymbol{A}_3\boldsymbol{B}_3 = \boldsymbol{I}_n \qquad \boldsymbol{A}_0\boldsymbol{B}_1 + \boldsymbol{A}_1\boldsymbol{B}_0 - \boldsymbol{A}_2\boldsymbol{B}_3 - \boldsymbol{A}_3\boldsymbol{B}_2 = \boldsymbol{0}$$
$$\boldsymbol{A}_0\boldsymbol{B}_2 - \boldsymbol{A}_1\boldsymbol{B}_3 + \boldsymbol{A}_2\boldsymbol{B}_0 - \boldsymbol{A}_3\boldsymbol{B}_1 = \boldsymbol{0} \qquad \boldsymbol{A}_0\boldsymbol{B}_3 + \boldsymbol{A}_1\boldsymbol{B}_2 + \boldsymbol{A}_2\boldsymbol{B}_1 + \boldsymbol{A}_3\boldsymbol{B}_0 = \boldsymbol{0}$$

$$\begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \\ -\boldsymbol{B}_1 & \boldsymbol{B}_0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ -\boldsymbol{B}_2 & -\boldsymbol{B}_3 & \boldsymbol{B}_0 & \boldsymbol{B}_1 \\ \boldsymbol{B}_3 & -\boldsymbol{B}_2 & -\boldsymbol{B}_1 & \boldsymbol{B}_0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_n & & & \\ & \boldsymbol{I}_n & & \\ & & \boldsymbol{I}_n & \\ & & & \boldsymbol{I}_n \end{pmatrix}$$

则

$$\begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \\ -\boldsymbol{B}_1 & \boldsymbol{B}_0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ -\boldsymbol{B}_2 & -\boldsymbol{B}_3 & \boldsymbol{B}_0 & \boldsymbol{B}_1 \\ \boldsymbol{B}_3 & -\boldsymbol{B}_2 & -\boldsymbol{B}_1 & \boldsymbol{B}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_n & & & \\ & \boldsymbol{I}_n & & \\ & & \boldsymbol{I}_n & \\ & & & \boldsymbol{I}_n \end{pmatrix}$$

所以：

$$\boldsymbol{B}_0\boldsymbol{A}_0 - \boldsymbol{B}_1\boldsymbol{A}_1 - \boldsymbol{B}_2\boldsymbol{A}_2 + \boldsymbol{B}_3\boldsymbol{A}_3 = \boldsymbol{I}_n \qquad \boldsymbol{B}_0\boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{A}_0 - \boldsymbol{B}_2\boldsymbol{A}_3 - \boldsymbol{B}_3\boldsymbol{A}_2 = \boldsymbol{0}$$

$$\boldsymbol{B}_0\boldsymbol{A}_2 - \boldsymbol{B}_1\boldsymbol{A}_3 + \boldsymbol{B}_2\boldsymbol{A}_0 - \boldsymbol{B}_3\boldsymbol{A}_1 = \boldsymbol{0} \qquad \boldsymbol{B}_0\boldsymbol{A}_3 + \boldsymbol{B}_1\boldsymbol{A}_2 + \boldsymbol{B}_2\boldsymbol{A}_1 + \boldsymbol{B}_3\boldsymbol{A}_0 = \boldsymbol{0}$$

故 $\boldsymbol{BA} = \boldsymbol{I}_{\mathrm{n}}$.

推论 1　设 $\boldsymbol{A}, \boldsymbol{B} \in M_{n\times n}(\boldsymbol{H})$，如果 $\boldsymbol{BA} = \boldsymbol{I}_n$，则 $\boldsymbol{AB} = \boldsymbol{I}_n$.

性质 2　设 $\boldsymbol{A}, \boldsymbol{B} \in M_{n\times n}(\boldsymbol{H})$，则：

(i) $\boldsymbol{I}_n^{\mathbb{R}} = \boldsymbol{I}_{4n}$；

(ii) $(\boldsymbol{A} + \boldsymbol{B})^{\mathbb{R}} = \boldsymbol{A}^{\mathbb{R}} + \boldsymbol{B}^{\mathbb{R}}$；

(iii) $(\boldsymbol{AB})^{\mathbb{R}} = \boldsymbol{A}^{\mathbb{R}}\boldsymbol{B}^{\mathbb{R}}$；

(iv) 若 $\boldsymbol{A}$ 可逆，则$(\boldsymbol{A}^{-1})^{\mathbb{R}} = (\boldsymbol{A}^{\mathbb{R}})^{-1}$.

证　由矩阵实表示的定义及矩阵的加法运算，(i) 和(ii) 成立.

(iii)　令

$$\boldsymbol{A}^{\mathbb{R}}\boldsymbol{B}^{\mathbb{R}} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \\ -\boldsymbol{B}_1 & \boldsymbol{B}_0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ -\boldsymbol{B}_2 & -\boldsymbol{B}_3 & \boldsymbol{B}_0 & \boldsymbol{B}_1 \\ \boldsymbol{B}_3 & -\boldsymbol{B}_2 & -\boldsymbol{B}_1 & \boldsymbol{B}_0 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}$$

则：

$$t_{11} = \boldsymbol{A}_0\boldsymbol{B}_0 - \boldsymbol{A}_1\boldsymbol{B}_1 - \boldsymbol{A}_2\boldsymbol{B}_2 + \boldsymbol{A}_3\boldsymbol{B}_3 \qquad t_{12} = \boldsymbol{A}_0\boldsymbol{B}_1 + \boldsymbol{A}_1\boldsymbol{B}_0 - \boldsymbol{A}_2\boldsymbol{B}_3 - \boldsymbol{A}_3\boldsymbol{B}_2$$

$$t_{13} = \boldsymbol{A}_0\boldsymbol{B}_2 - \boldsymbol{A}_1\boldsymbol{B}_3 + \boldsymbol{A}_2\boldsymbol{B}_0 - \boldsymbol{A}_3\boldsymbol{B}_1 \qquad t_{14} = \boldsymbol{A}_0\boldsymbol{B}_3 + \boldsymbol{A}_1\boldsymbol{B}_2 + \boldsymbol{A}_2\boldsymbol{B}_1 + \boldsymbol{A}_3\boldsymbol{B}_0$$

$$t_{21} = -\boldsymbol{A}_1\boldsymbol{B}_0 - \boldsymbol{A}_0\boldsymbol{B}_1 + \boldsymbol{A}_3\boldsymbol{B}_2 + \boldsymbol{A}_2\boldsymbol{B}_3 \qquad t_{22} = -\boldsymbol{A}_1\boldsymbol{B}_1 + \boldsymbol{A}_0\boldsymbol{B}_0 + \boldsymbol{A}_3\boldsymbol{B}_3 - \boldsymbol{A}_2\boldsymbol{B}_2$$

$$t_{23} = -\boldsymbol{A}_1\boldsymbol{B}_2 - \boldsymbol{A}_0\boldsymbol{B}_3 - \boldsymbol{A}_3\boldsymbol{B}_0 - \boldsymbol{A}_2\boldsymbol{B}_1 \qquad t_{24} = -\boldsymbol{A}_1\boldsymbol{B}_3 + \boldsymbol{A}_0\boldsymbol{B}_2 - \boldsymbol{A}_3\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{B}_0$$

$$t_{31} = -\boldsymbol{A}_2\boldsymbol{B}_0 + \boldsymbol{A}_3\boldsymbol{B}_1 - \boldsymbol{A}_0\boldsymbol{B}_2 + \boldsymbol{A}_1\boldsymbol{B}_3 \qquad t_{32} = -\boldsymbol{A}_2\boldsymbol{B}_1 - \boldsymbol{A}_3\boldsymbol{B}_0 - \boldsymbol{A}_0\boldsymbol{B}_3 - \boldsymbol{A}_1\boldsymbol{B}_2$$

$$t_{33} = -\boldsymbol{A}_2\boldsymbol{B}_2 + \boldsymbol{A}_3\boldsymbol{B}_3 + \boldsymbol{A}_0\boldsymbol{B}_0 - \boldsymbol{A}_1\boldsymbol{B}_1 \qquad t_{34} = -\boldsymbol{A}_2\boldsymbol{B}_3 - \boldsymbol{A}_3\boldsymbol{B}_2 + \boldsymbol{A}_0\boldsymbol{B}_1 + \boldsymbol{A}_1\boldsymbol{B}_0$$

$$t_{41} = \boldsymbol{A}_3\boldsymbol{B}_0 + \boldsymbol{A}_2\boldsymbol{B}_1 + \boldsymbol{A}_1\boldsymbol{B}_2 + \boldsymbol{A}_0\boldsymbol{B}_3 \qquad t_{42} = \boldsymbol{A}_3\boldsymbol{B}_1 - \boldsymbol{A}_2\boldsymbol{B}_0 + \boldsymbol{A}_1\boldsymbol{B}_3 - \boldsymbol{A}_0\boldsymbol{B}_2$$

$$t_{43} = \boldsymbol{A}_3\boldsymbol{B}_2 + \boldsymbol{A}_2\boldsymbol{B}_3 - \boldsymbol{A}_1\boldsymbol{B}_0 - \boldsymbol{A}_0\boldsymbol{B}_1 \qquad t_{44} = \boldsymbol{A}_3\boldsymbol{B}_3 - \boldsymbol{A}_2\boldsymbol{B}_2 - \boldsymbol{A}_1\boldsymbol{B}_1 + \boldsymbol{A}_0\boldsymbol{B}_0$$

令

$$(\boldsymbol{AB})^{\mathbb{R}} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix}$$

则：

$$u_{11} = t_{11} \qquad u_{12} = t_{12} \qquad u_{13} = t_{13} \qquad u_{14} = t_{14}$$

$$u_{21} = t_{21} \qquad u_{22} = t_{22} \qquad u_{23} = t_{23} \qquad u_{24} = t_{24}$$

$$u_{31} = t_{31} \qquad u_{32} = t_{32} \qquad u_{33} = t_{33} \qquad u_{34} = t_{34}$$

$$u_{41} = t_{41} \qquad u_{42} = t_{42} \qquad u_{43} = t_{43} \qquad u_{44} = t_{44}$$

故$(\boldsymbol{AB})^{\mathbb{R}} = \boldsymbol{A}^{\mathbb{R}}\boldsymbol{B}^{\mathbb{R}}$.

(iv)　由 $\boldsymbol{A}$ 可逆及(iii) 知，$(\boldsymbol{AA}^{-1})^{\mathbb{R}} = \boldsymbol{A}^{\mathbb{R}}(\boldsymbol{A}^{-1})^{\mathbb{R}} = \boldsymbol{I}_{4n}$，故 $(\boldsymbol{A}^{-1})^{\mathbb{R}} = (\boldsymbol{A}^{\mathbb{R}})^{-1}$.

定义 2[12]　设 $\boldsymbol{A} \in M_{n\times n}(\boldsymbol{H})$，$\lambda \in \boldsymbol{H}$，对于非零 n 维列向量 $\boldsymbol{x}$，如果 $\boldsymbol{Ax} = \lambda\boldsymbol{x}$，则称 λ 为 $\boldsymbol{A}$ 的特征值，$\boldsymbol{x}$

为属于 λ 的特征向量. $\boldsymbol{A}$ 的所有特征值的集合 $\sigma(\boldsymbol{A})=\{\lambda\in\boldsymbol{H}:\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x},\ \boldsymbol{x}\neq\boldsymbol{0}\}$ 为 $\boldsymbol{A}$ 的谱.

定理 2 设 $\boldsymbol{A}\in M_{n\times n}(\boldsymbol{H})$, $\boldsymbol{A}^{\mathrm{T}}$ 表示 $\boldsymbol{A}$ 的转置, 则 $\boldsymbol{A}$ 有 $2n$ 个复特征值.

证 令

$$\boldsymbol{A}=\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3 \qquad \boldsymbol{A}_0,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3\in M_{n\times n}(\mathbb{R})$$

$$\boldsymbol{x}=\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3$$

由 $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$, 则

$$(\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3)(\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3)=\lambda(\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3)$$

展开得:

$$\boldsymbol{A}_0\boldsymbol{x}_0-\boldsymbol{A}_1\boldsymbol{x}_1-\boldsymbol{A}_2\boldsymbol{x}_2+\boldsymbol{A}_3\boldsymbol{x}_3=\lambda\boldsymbol{x}_0 \qquad \boldsymbol{A}_0\boldsymbol{x}_1+\boldsymbol{A}_1\boldsymbol{x}_0-\boldsymbol{A}_2\boldsymbol{x}_3-\boldsymbol{A}_3\boldsymbol{x}_2=\lambda\boldsymbol{x}_1$$

$$\boldsymbol{A}_0\boldsymbol{x}_2-\boldsymbol{A}_1\boldsymbol{x}_3+\boldsymbol{A}_2\boldsymbol{x}_0-\boldsymbol{A}_3\boldsymbol{x}_1=\lambda\boldsymbol{x}_2 \qquad \boldsymbol{A}_0\boldsymbol{x}_3+\boldsymbol{A}_1\boldsymbol{x}_2+\boldsymbol{A}_2\boldsymbol{x}_1+\boldsymbol{A}_3\boldsymbol{x}_0=\lambda\boldsymbol{x}_3$$

即

$$\begin{pmatrix}\boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3\\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2\\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1\\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}_0\\ \boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\end{pmatrix}=\lambda\begin{pmatrix}\boldsymbol{x}_0\\ \boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\end{pmatrix}$$

故 $\boldsymbol{A}_{n\times n}$ 有 $2n$ 个复特征值.

推论 2 记 $\boldsymbol{A}^{\mathbb{R}}$ 的谱 $\sigma(\boldsymbol{A}^{\mathbb{R}})=\{\lambda\in\mathbb{C}:\boldsymbol{A}^{\mathbb{R}}\boldsymbol{v}=\lambda\boldsymbol{v},\ \boldsymbol{v}\neq\boldsymbol{0}\}$, $\mathbb{C}$ 为复数集, $\boldsymbol{A}\in M_{n\times n}(\boldsymbol{H})$, 则 $\sigma(\boldsymbol{A})\cap\mathbb{C}=\sigma(\boldsymbol{A}^{\mathbb{R}})$.

定理 3 设 $\boldsymbol{A}=\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3\in M_{n\times n}(\boldsymbol{H})$, 则 $\lambda=\lambda_0+i\lambda_1+j\lambda_2+k\lambda_3$ 为 $\boldsymbol{A}$ 的特征值当且仅当存在非零的实 n 维列向量 $\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3$, 满足

$$\begin{pmatrix}\boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n & -\boldsymbol{A}_1+\lambda_1\boldsymbol{I}_n & -\boldsymbol{A}_2+\lambda_2\boldsymbol{I}_n & \boldsymbol{A}_3-\lambda_3\boldsymbol{I}_n\\ \boldsymbol{A}_1-\lambda_1\boldsymbol{I}_n & \boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n & -\boldsymbol{A}_3+\lambda_3\boldsymbol{I}_n & -\boldsymbol{A}_2+\lambda_2\boldsymbol{I}_n\\ \boldsymbol{A}_2-\lambda_2\boldsymbol{I}_n & -\boldsymbol{A}_3+\lambda_3\boldsymbol{I}_n & \boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n & -\boldsymbol{A}_1+\lambda_1\boldsymbol{I}_n\\ \boldsymbol{A}_3-\lambda_3\boldsymbol{I}_n & \boldsymbol{A}_2-\lambda_2\boldsymbol{I}_n & \boldsymbol{A}_1-\lambda_1\boldsymbol{I}_n & \boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_0\\ \boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\end{pmatrix}=\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}$$

证 由 $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$ 以及

$$(\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3)(\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3)=(\lambda_0+i\lambda_1+j\lambda_2+k\lambda_3)(\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3)$$

则:

$$(\boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n)\boldsymbol{x}_0+(-\boldsymbol{A}_1+\lambda_1\boldsymbol{I}_n)\boldsymbol{x}_1+(-\boldsymbol{A}_2+\lambda_2\boldsymbol{I}_n)\boldsymbol{x}_2+(\boldsymbol{A}_3-\lambda_3\boldsymbol{I}_n)\boldsymbol{x}_3=\boldsymbol{0}$$

$$(\boldsymbol{A}_1-\lambda_1\boldsymbol{I}_n)\boldsymbol{x}_0+(\boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n)\boldsymbol{x}_1+(-\boldsymbol{A}_3+\lambda_3\boldsymbol{I}_n)\boldsymbol{x}_2+(-\boldsymbol{A}_2+\lambda_2\boldsymbol{I}_n)\boldsymbol{x}_3=\boldsymbol{0}$$

$$(\boldsymbol{A}_2-\lambda_2\boldsymbol{I}_n)\boldsymbol{x}_0+(-\boldsymbol{A}_3+\lambda_3\boldsymbol{I}_n)\boldsymbol{x}_1+(\boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n)\boldsymbol{x}_2+(-\boldsymbol{A}_1+\lambda_1\boldsymbol{I}_n)\boldsymbol{x}_3=\boldsymbol{0}$$

$$(\boldsymbol{A}_3-\lambda_3\boldsymbol{I}_n)\boldsymbol{x}_0+(\boldsymbol{A}_2-\lambda_2\boldsymbol{I}_n)\boldsymbol{x}_1+(\boldsymbol{A}_1-\lambda_1\boldsymbol{I}_n)\boldsymbol{x}_2+(\boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n)\boldsymbol{x}_3=\boldsymbol{0}$$

故

$$\begin{pmatrix}\boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n & -\boldsymbol{A}_1+\lambda_1\boldsymbol{I}_n & -\boldsymbol{A}_2+\lambda_2\boldsymbol{I}_n & \boldsymbol{A}_3-\lambda_3\boldsymbol{I}_n\\ \boldsymbol{A}_1-\lambda_1\boldsymbol{I}_n & \boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n & -\boldsymbol{A}_3+\lambda_3\boldsymbol{I}_n & -\boldsymbol{A}_2+\lambda_2\boldsymbol{I}_n\\ \boldsymbol{A}_2-\lambda_2\boldsymbol{I}_n & -\boldsymbol{A}_3+\lambda_3\boldsymbol{I}_n & \boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n & -\boldsymbol{A}_1+\lambda_1\boldsymbol{I}_n\\ \boldsymbol{A}_3-\lambda_3\boldsymbol{I}_n & \boldsymbol{A}_2-\lambda_2\boldsymbol{I}_n & \boldsymbol{A}_1-\lambda_1\boldsymbol{I}_n & \boldsymbol{A}_0-\lambda_0\boldsymbol{I}_n\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_0\\ \boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\end{pmatrix}=\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}$$

定义 3[12] 设 $\boldsymbol{A},\boldsymbol{B}\in M_{n\times n}(\boldsymbol{H})$, 如果 $\boldsymbol{A}\boldsymbol{B}=\boldsymbol{B}\boldsymbol{A}=\boldsymbol{I}_n$, 则称矩阵 $\boldsymbol{A}$ 可逆, 且 $\boldsymbol{A}^{-1}=\boldsymbol{B}$.

定理 4 设 $\boldsymbol{A}=\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3\in M_{n\times n}(\boldsymbol{H})$, 则以下命题等价:

(i) 方程 $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$ 有唯一解;

(ii) $\boldsymbol{A}^{\mathbb{R}}$ 可逆;

(iii) $\boldsymbol{A}$ 可逆;

(iv) $\boldsymbol{A}$ 没有零特征值.

证 (i)⇒(ii) 因

$$\boldsymbol{A}=\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3 \qquad \boldsymbol{x}=\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3$$

由：

$$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0} \qquad (\boldsymbol{A}_0+i\boldsymbol{A}_1+j\boldsymbol{A}_2+k\boldsymbol{A}_3)(\boldsymbol{x}_0+i\boldsymbol{x}_1+j\boldsymbol{x}_2+k\boldsymbol{x}_3)=\boldsymbol{0}$$

故：

$$\boldsymbol{A}_0\boldsymbol{x}_0-\boldsymbol{A}_1\boldsymbol{x}_1-\boldsymbol{A}_2\boldsymbol{x}_2+\boldsymbol{A}_3\boldsymbol{x}_3=\boldsymbol{0} \qquad \boldsymbol{A}_0\boldsymbol{x}_1+\boldsymbol{A}_1\boldsymbol{x}_0-\boldsymbol{A}_2\boldsymbol{x}_3-\boldsymbol{A}_3\boldsymbol{x}_2=\boldsymbol{0}$$

$$\boldsymbol{A}_0\boldsymbol{x}_2-\boldsymbol{A}_1\boldsymbol{x}_3+\boldsymbol{A}_2\boldsymbol{x}_0-\boldsymbol{A}_3\boldsymbol{x}_1=\boldsymbol{0} \qquad \boldsymbol{A}_0\boldsymbol{x}_3+\boldsymbol{A}_1\boldsymbol{x}_2+\boldsymbol{A}_2\boldsymbol{x}_1+\boldsymbol{A}_3\boldsymbol{x}_0=\boldsymbol{0}$$

则 $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$ 的充要条件为

$$\begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$$

即$(\boldsymbol{A}^{\mathbb{R}})^{\mathrm{T}}(\boldsymbol{x}_0 \quad \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3)^{\mathrm{T}}=\boldsymbol{0}$. 由 $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$ 有唯一解，故$(\boldsymbol{A}^{\mathbb{R}})^{\mathrm{T}}(\boldsymbol{x}_0 \quad \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3)^{\mathrm{T}}=\boldsymbol{0}$ 有唯一解，$\boldsymbol{A}^{\mathbb{R}}$ 可逆.

(ii)⇒(iii)　由 $\boldsymbol{A}^{\mathbb{R}}$ 可逆，则存在 $\boldsymbol{B}^{\mathbb{R}}$，使得

$$\begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \\ -\boldsymbol{B}_1 & \boldsymbol{B}_0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ -\boldsymbol{B}_2 & -\boldsymbol{B}_3 & \boldsymbol{B}_0 & \boldsymbol{B}_1 \\ \boldsymbol{B}_3 & -\boldsymbol{B}_2 & -\boldsymbol{B}_1 & \boldsymbol{B}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ -\boldsymbol{A}_1 & \boldsymbol{A}_0 & -\boldsymbol{A}_3 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & -\boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_3 & -\boldsymbol{A}_2 & -\boldsymbol{A}_1 & \boldsymbol{A}_0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_n & & & \\ & \boldsymbol{I}_n & & \\ & & \boldsymbol{I}_n & \\ & & & \boldsymbol{I}_n \end{pmatrix}$$

故：

$$\boldsymbol{B}_0\boldsymbol{A}_0-\boldsymbol{B}_1\boldsymbol{A}_1-\boldsymbol{B}_2\boldsymbol{A}_2+\boldsymbol{B}_3\boldsymbol{A}_3=\boldsymbol{I}_n \qquad \boldsymbol{B}_0\boldsymbol{A}_1+\boldsymbol{B}_1\boldsymbol{A}_0-\boldsymbol{B}_2\boldsymbol{A}_3-\boldsymbol{B}_3\boldsymbol{A}_2=\boldsymbol{0}$$

$$\boldsymbol{B}_0\boldsymbol{A}_2-\boldsymbol{B}_1\boldsymbol{A}_3+\boldsymbol{B}_2\boldsymbol{A}_0-\boldsymbol{B}_3\boldsymbol{A}_1=\boldsymbol{0} \qquad \boldsymbol{B}_0\boldsymbol{A}_3+\boldsymbol{B}_1\boldsymbol{A}_2+\boldsymbol{B}_2\boldsymbol{A}_1+\boldsymbol{B}_3\boldsymbol{A}_0=\boldsymbol{0}$$

所以 $\boldsymbol{B}\boldsymbol{A}=\boldsymbol{I}_n$，再由推论 1，$\boldsymbol{A}\boldsymbol{B}=\boldsymbol{I}_n$，$\boldsymbol{A}$ 可逆.

由(iii) 推出(i) 和(iv) 的证明是平凡的，不赘述.

对于混合型交换四元数矩阵，有下面的盖尔圆盘定理成立：

定理 5　设 $\boldsymbol{A}=(a_{ij})\in M_{n\times n}(\boldsymbol{H})$，则 $\sigma(\boldsymbol{A})\subseteq\bigcup_{i=1}^{n}\left\{h\in\boldsymbol{H}:\ \|h-a_{ii}\|\leqslant\sum_{j=1,\ j\neq i}^{n}\|a_{ij}\|\right\}$.

证　令 $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$，$\lambda$ 为 $\boldsymbol{A}$ 的特征值，$\boldsymbol{x}$ 为属于 λ 的特征向量. 假设 $\boldsymbol{x}$ 的第 i 个分量 $\boldsymbol{x}_i$ 满足

$$\|\boldsymbol{x}_i\|\geqslant\|\boldsymbol{x}_j\| \qquad j=1,2,\cdots,n$$

则 $\|\boldsymbol{x}_i\|\geqslant 0$. $\boldsymbol{A}\boldsymbol{x}$ 的第 i 个分量为 $\lambda\boldsymbol{x}_i$，即 $\lambda\boldsymbol{x}_i=\sum_{j=1}^{n}a_{ij}\boldsymbol{x}_j$. 则

$$(\lambda-a_{ii})\boldsymbol{x}_i=\sum_{j=1,\ j\neq i}^{n}a_{ij}\boldsymbol{x}_j$$

取范数：

$$\|(\lambda-a_{ii})\boldsymbol{x}_i\|=\left\|\sum_{j=1,\ j\neq i}^{n}a_{ij}\boldsymbol{x}_j\right\|$$

$$\|(\lambda-a_{ii})\|\,\|\boldsymbol{x}_i\|\leqslant\sum_{j=1,\ j\neq i}^{n}\|a_{ij}\boldsymbol{x}_j\|\leqslant\sum_{j=1,\ j\neq i}^{n}\|a_{ij}\|\,\|\boldsymbol{x}_i\|$$

故

$$\|(\lambda-a_{ii})\|\leqslant\sum_{j=1,\ j\neq i}^{n}\|a_{ij}\|$$

则

$$\sigma(A)\subseteq\bigcup_{i=1}^{n}\left\{h\in H:\ \|h-a_{ii}\|\leqslant\sum_{j=1,\ j\neq i}^{n}\|a_{ij}\|\right\}$$

3　数值例子

设

$$\boldsymbol{A}=\begin{pmatrix}-i & 1\\ 1+j+k & i\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+i\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}+j\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+k\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\in M_{2\times 2}(\boldsymbol{H})$$

则:

$$\boldsymbol{A}^{\mathbb{R}}=\begin{pmatrix}0 & 1 & -1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\ 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0\end{pmatrix}$$

$$\sigma(\boldsymbol{A}^{\mathbb{R}})=\{-1.0987\pm 0.4551\mathrm{i},-0.4551\pm 1.0987\mathrm{i},1.0987\pm 0.4551\mathrm{i},0.4551\pm 1.0987\mathrm{i}\}$$

盖尔圆盘:

$$T_1=\{k\in\mathbb{C}:\ \|k+\mathrm{i}\|\leqslant 1\}\qquad T_2=\{k\in\mathbb{C}:\ \|k-\mathrm{i}\|\leqslant\sqrt[4]{5}\approx 1.4953\}$$

故

$$\sigma(\boldsymbol{A})\cap\mathbb{C}\subseteq T_1\cup T_2$$

经检验知,所得结论正确.

4 结 语

对非交换四元数、分裂四元数及其矩阵的研究涉及的内容十分广泛,包括实表示、复表示、逆矩阵、相似性、对角化、特征值和特征向量、矩阵分解、最小二乘[13] 等问题,并已取得丰硕成果.在这些研究成果的基础上,可将混合型交换四元数矩阵的研究进一步推广到矩阵方程中[14-15].

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On Property and Application of Real Representation of a Mixed Type Commutative Quaternion Matrix

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Abstract：In this paper，studies have been done on the basis of introducing the concept of mixed type commutative quaternion and mixed type commutative quaternion matrix. Firstly，it is proved that the mixed type commutative quaternion and the fourth-order matrix on the real field are isomorphic. The study of mixed type commutative quaternion is transformed into the study of the fourth-order matrix on the real field. Secondly，on the basis of the isomorphism of the $4n$ th-order matrix in the real field and the mixed type commutative quaternion matrix，the study of mixed type commutative quaternion matrix is transformed into the study of the $4n$ th-order matrix on the real field. By means of the properties of real matrices to obtain the series properties of the real representation of mixed type commutative quaternion matrices，and the equivalent condition for the reversibility of the mixed type commutative quaternion matrix is given. Based on the property of the real representation of mixed type commutative quaternion matrix，the number of complex eigenvalues of mixed type commutative quaternion matrix and the sufficient and necessary conditions for the existence of eigenvalues are obtained. The Gerschgorin disk theorem on the real field is extended to the mixed type commutative quaternion matrix. Finally，the correctness and validity of the Gerschgorin disk theorem for mixed type commutative quaternion matrix is verified by a numerical example.

Key words：mixed type commutative quaternion matrix；real representation；matrix eigenvalue；Gerschgorin disk theorem

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