DOI:10.13718/j.cnki.xsxb.2019.06.008

双体系统中保持 von Neumann 熵的量子信道的结构[①]

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摘要: 设 H_m 是维数为 m 的复希尔伯特空间,$S(H_m \otimes H_n)$ 是作用在复双体希尔伯特空间 $H_m \otimes H_n$ 上的所有量子态的全体,$S_{sep}(H_m \otimes H_n)$ 是所有可分量子态做成的 $S(H_m \otimes H_n)$ 的凸子集,ϕ: $S(H_m \otimes H_n) \longrightarrow S(H_m \otimes H_n)$ 是量子信道且 $\phi(S_{sep}(H_m \otimes H_n)) = S_{sep}(H_m \otimes H_n)$,那么 ϕ 保持 von Neumann 熵 $S(t\boldsymbol{\rho} + (1-t)\boldsymbol{\sigma}) = S(t\phi(\boldsymbol{\rho}) + (1-t)\phi(\boldsymbol{\sigma}))$,$\forall\, t \in [0, 1]$,$\forall\, \boldsymbol{\rho}, \boldsymbol{\sigma} \in S_{sep}(H_m \otimes H_n)$ 当且仅当在 H_m, H_n 上分别存在酉算子或共轭酉算子 $\overline{\boldsymbol{U}}_m, \overline{\boldsymbol{V}}_n$,使得 $\phi(\boldsymbol{\rho}) = (\overline{\boldsymbol{U}}_m \otimes \overline{\boldsymbol{V}}_n)\boldsymbol{\rho}(\overline{\boldsymbol{U}}_m \otimes \overline{\boldsymbol{V}}_n)^*$,$\forall\, \boldsymbol{\rho} \in S_{sep}(H_m \otimes H_n)$.

关 键 词: 量子信道;von Neumann 熵;量子态

中图分类号: O177.1 **文献标志码:** A **文章编号:** 1000-5471(2019)06-0031-06

量子态叫作密度矩阵,是作用在复希尔伯特空间上的半正定迹 1 矩阵. 量子态 $\boldsymbol{\rho}$ 是纯态当且仅当 $\boldsymbol{\rho}^2 = \boldsymbol{\rho}$,即 $\boldsymbol{\rho}$ 是秩 1 投影. 若 $\boldsymbol{\rho}^2 \neq \boldsymbol{\rho}$,则 $\boldsymbol{\rho}$ 是混合态. 复希尔伯特空间 H 上的所有量子态记为 $S(H)$,它是个凸集,所有纯态记为 $\mathrm{Pur}(H)$,显然 $\mathrm{Pur}(H)$ 是 $S(H)$ 的子集. 在量子信息理论中,$H = H_1 \otimes H_2$ 叫双体系统,其中 H_1 和 H_2 都是有限维复希尔伯特空间. 对于 $\boldsymbol{\rho} \in S(H_1 \otimes H_2)$,如果 $\boldsymbol{\rho}$ 可以写成 $\boldsymbol{\rho} = \sum_{i=1}^{n} p_i \boldsymbol{\rho}_i \otimes \boldsymbol{\sigma}_i$,其中 $\boldsymbol{\rho}_i \in S(H_1)$,$\boldsymbol{\sigma}_i \in S(H_2)$,$\sum_{i=1}^{n} p_i = 1$,$p_i \geqslant 0$,则这时就说量子态 $\boldsymbol{\rho}$ 可分,否则量子态 $\boldsymbol{\rho}$ 就是纠缠的. 下面分别用符号 $S_{sep}(H_m \otimes H_n)$ 和 $\mathrm{Pur}_{sep}(H_m \otimes H_n)$ 表示双体系统 $H_m \otimes H_n$ 上的所有可分量子态和可分量子纯态. $M(H_n)$ 表示所有 n 阶方阵,对于 $\boldsymbol{A} \in M(H_n)$,用 $\boldsymbol{A}^*$ 表示矩阵 $\boldsymbol{A}$ 的共轭转置.

文献[1]对量子信息科学的线性保持问题作了一个概述. 很快,文献[2]也找到了保持 KY FAN 范数和 SCHATTEN 范数不变的矩阵张量积之间的线性变换的结构形式. 文献[3]也研究了相同的定义和问题. 接着,文献[4]把文献[3]的结果推广到无限维希尔伯特空间. 这些成果研究的算子张量积之间的映射都是线性的. 文献[5-6]研究的是关于量子测量(如冯诺依曼熵、Tsallis 熵)单体系统上的非线性映射. 文献[7]刻画了一个作用在双体量子系统 $H_1 \otimes H_2$ 里的所有可分态上并保持凸组合的双射的结构. 现在的目的就是借助 von Neuamnn 熵的性质来研究一个保持 von Neumann 熵量子信道的结构. 对于 $\boldsymbol{\rho} \in S(H)$,如果 λ_i 是 $\boldsymbol{\rho}$ 的特征值,von Neumann 熵 $S(\boldsymbol{\rho})$ 的定义如下[8]:$S(\boldsymbol{\rho}) = -\sum_i \lambda_i \log(\lambda_i)$,$\lambda_i \geqslant 0$. 规定 $0\log 0 = 0$,对数的底数通常取 2. 一个量子信道是保迹完全正线性映射 Φ: $M(H_m) \longrightarrow M(H_n)$ 当且仅当有表达式 $\Phi(\boldsymbol{X}) = \sum_i \boldsymbol{A}_i \boldsymbol{X} \boldsymbol{A}_i^*$,其中 $\boldsymbol{A}_i$ 是 $m \times n$ 阶矩阵、$\sum_i \boldsymbol{A}_i^* \boldsymbol{A}_i = \mathrm{id}_m$. 量子信道总是仿射. 近年来有关量子熵、量子相位门的研究可见文献[9-13]. 下面是本文的主要结果:

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① 收稿日期:2018-07-23

基金项目:国家自然科学基金项目(11871375).

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定理 1 设 H_m,H_n 分别是维数为 m,n 的复希尔伯特空间，ϕ：$S(H_m \otimes H_n) \longrightarrow S(H_m \otimes H_n)$ 是量子信道，且 $\phi(S_{sep}(H_m \otimes H_n)) = S_{sep}(H_m \otimes H_n)$. 那么对 $\forall t \in [0,1]$ 和 $\forall \boldsymbol{\rho},\boldsymbol{\sigma} \in S_{sep}(H_m \otimes H_n)$，$S(t\boldsymbol{\rho} + (1-t)\boldsymbol{\sigma}) = S(t\phi(\boldsymbol{\rho}) + (1-t)\phi(\boldsymbol{\sigma}))$ 当且仅当在 H_m,H_n 上分别存在酉矩阵或共轭酉矩阵 $\overline{\boldsymbol{U}}_m,\overline{\boldsymbol{V}}_n$，使得 $\phi(\boldsymbol{\rho}) = (\overline{\boldsymbol{U}}_m \otimes \overline{\boldsymbol{V}}_n)\boldsymbol{\rho}\,(\overline{\boldsymbol{U}}_m \otimes \overline{\boldsymbol{V}}_n)^*$，$\forall \boldsymbol{\rho} \in S_{sep}(H_m \otimes H_n)$.

对于 $\boldsymbol{P} \in \mathrm{Pur}(H_m)$ 和 $\boldsymbol{Q} \in \mathrm{Pur}(H_n)$，分别记：

$$L_{\boldsymbol{P}} = \{\boldsymbol{P} \otimes \boldsymbol{Q}_1 \in S(H_m \otimes H_n): \forall \boldsymbol{Q}_1 \in \mathrm{Pur}(H_n)\}$$

$$R_{\boldsymbol{Q}} = \{\boldsymbol{P}_1 \otimes \boldsymbol{Q} \in S(H_m \otimes H_n): \forall \boldsymbol{P}_1 \in \mathrm{Pur}(H_m)\}$$

引理 1 设 H_m,H_n 分别是维数为 m,n 的复希尔伯特空间，ϕ：$S(H_m \otimes H_n) \longrightarrow S(H_m \otimes H_n)$ 是量子信道，且 $\phi(S_{sep}(H_m \otimes H_n)) = S_{sep}(H_m \otimes H_n)$. 如果对 $\forall \boldsymbol{\rho},\boldsymbol{\sigma} \in S_{sep}(H_m \otimes H_n)$ 和 $\forall t \in [0,1]$，有 $S(t\boldsymbol{\rho} + (1-t)\boldsymbol{\sigma}) = S(t\phi(\boldsymbol{\rho}) + (1-t)\phi(\boldsymbol{\sigma}))$，那么下面结论之一成立：

(i) 对 $\forall \boldsymbol{P} \in \mathrm{Pur}(H_m)$，至少存在一个 $\overline{\boldsymbol{P}} \in \mathrm{Pur}(H_m)$，使得 $\phi(L_{\boldsymbol{P}}) \subseteq L_{\overline{\boldsymbol{P}}}$；

(ii) 对 $\forall \boldsymbol{P} \in \mathrm{Pur}(H_m)$，至少存在一个 $\overline{\boldsymbol{Q}} \in \mathrm{Pur}(H_n)$，使得 $\phi(L_{\boldsymbol{P}}) \subseteq R_{\overline{\boldsymbol{Q}}}$.

证 由文献[5]知 $\phi(\mathrm{Pur}_{sep}(H_m \otimes H_n)) = \mathrm{Pur}_{sep}(H_m \otimes H_n)$. 于是对于任意可分纯态 $\boldsymbol{P} \otimes \boldsymbol{Q} \in \mathrm{Pur}_{sep}(H_m \otimes H_n)$，都至少存在一个可分纯态 $\overline{\boldsymbol{P}} \otimes \overline{\boldsymbol{Q}} \in \mathrm{Pur}_{sep}(H_m \otimes H_n)$，使得 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}) = \overline{\boldsymbol{P}} \otimes \overline{\boldsymbol{Q}}$.

首先固定 $\boldsymbol{P} \in \mathrm{Pur}(H_m)$，对 $\forall \boldsymbol{Q}_1,\boldsymbol{Q}_2 \in \mathrm{Pur}(H_n)(\boldsymbol{Q}_1 \neq \boldsymbol{Q}_2)$，由文献[5]知，存在纯态 $\overline{\boldsymbol{P}}_i \in \mathrm{Pur}(H_m)$，$\overline{\boldsymbol{Q}}_i \in \mathrm{Pur}(H_n)(i=1,2)$，使得 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}_1) = \overline{\boldsymbol{P}}_1 \otimes \overline{\boldsymbol{Q}}_1$ 和 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}_2) = \overline{\boldsymbol{P}}_2 \otimes \overline{\boldsymbol{Q}}_2$.

因为量子信道总是作用在量子态上的仿射，即 ϕ 是双射，且

$$\phi(t\boldsymbol{\rho} + (1-t)\boldsymbol{\sigma}) = t\phi(\boldsymbol{\rho}) + (1-t)\phi(\boldsymbol{\sigma}) \qquad \forall \boldsymbol{\rho},\boldsymbol{\sigma} \in S(H_m),\ t \in [0,1]$$

则如果 $\boldsymbol{P} \otimes \boldsymbol{Q}_1 \neq \boldsymbol{P} \otimes \boldsymbol{Q}_2$，有 $\overline{\boldsymbol{P}}_1 \otimes \overline{\boldsymbol{Q}}_1 \neq \overline{\boldsymbol{P}}_2 \otimes \overline{\boldsymbol{Q}}_2$，而且

$$\overline{\boldsymbol{P}}_{t_1} \otimes \overline{\boldsymbol{Q}}_{t_1} = \phi(\boldsymbol{P} \otimes (t\boldsymbol{Q}_1 + (1-t)\boldsymbol{Q}_2)) = \phi(t\boldsymbol{P} \otimes \boldsymbol{Q}_1 + (1-t)\boldsymbol{P} \otimes \boldsymbol{Q}_2) = t\,\overline{\boldsymbol{P}}_1 \otimes \overline{\boldsymbol{Q}}_1 + (1-t)\,\overline{\boldsymbol{P}}_2 \otimes \overline{\boldsymbol{Q}}_2$$

根据文献[7]的引理 2 知 $\overline{\boldsymbol{Q}}_1$ 与 $\overline{\boldsymbol{Q}}_2$ 线性相关，或者 $\overline{\boldsymbol{P}}_1$ 与 $\overline{\boldsymbol{P}}_2$ 线性相关.

若 $\overline{\boldsymbol{Q}}_1$ 与 $\overline{\boldsymbol{Q}}_2$ 线性相关，因为 $\overline{\boldsymbol{P}}_i$ 与 $\overline{\boldsymbol{Q}}_i$ 都是纯态，而且 $\overline{\boldsymbol{P}}_1 \otimes \overline{\boldsymbol{Q}}_1 \neq \overline{\boldsymbol{P}}_2 \otimes \overline{\boldsymbol{Q}}_2$，所以 $\overline{\boldsymbol{Q}}_1 = \overline{\boldsymbol{Q}}_2$，且 $\overline{\boldsymbol{P}}_1,\overline{\boldsymbol{P}}_2$ 线性无关. 记 $\overline{\boldsymbol{Q}}_1 = \overline{\boldsymbol{Q}}_2 = \overline{\boldsymbol{Q}}$，则 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}_1) = \overline{\boldsymbol{P}}_1 \otimes \overline{\boldsymbol{Q}}$ 且 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}_2) = \overline{\boldsymbol{P}}_2 \otimes \overline{\boldsymbol{Q}}$. 现在，对 $\forall \boldsymbol{Q} \in \mathrm{Pur}(H_n)$，设 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}) = \overline{\boldsymbol{P}} \otimes \overline{\overline{\boldsymbol{Q}}}$，这时 $\overline{\boldsymbol{P}}$ 至少与 $\overline{\boldsymbol{P}}_1,\overline{\boldsymbol{P}}_2$ 其中一个线性无关，否则将会出现 $\overline{\boldsymbol{P}}_1 = \overline{\boldsymbol{P}}_2$. 不妨设 $\overline{\boldsymbol{P}}$ 与 $\overline{\boldsymbol{P}}_1$ 线性无关. 注意到文献[7]的引理 2 和等式

$$\overline{\boldsymbol{P}}_{t_2} \otimes \overline{\boldsymbol{Q}}_{t_2} = \phi(\boldsymbol{P} \otimes (t\boldsymbol{Q}_1 + (1-t)\boldsymbol{Q})) = \phi(t\boldsymbol{P} \otimes \boldsymbol{Q}_1 + (1-t)\boldsymbol{P} \otimes \boldsymbol{Q}) = t\,\overline{\boldsymbol{P}}_1 \otimes \overline{\boldsymbol{Q}} + (1-t)\,\overline{\boldsymbol{P}} \otimes \overline{\overline{\boldsymbol{Q}}}$$

得到 $\overline{\boldsymbol{Q}} = \overline{\overline{\boldsymbol{Q}}}$. 于是对于某一固定的 $\boldsymbol{P} \in \mathrm{Pur}(H_m)$，存在 $\overline{\boldsymbol{Q}} \in \mathrm{Pur}(H_n)$，使得 $\phi(\boldsymbol{P} \otimes \boldsymbol{Q}) \in R_{\overline{\boldsymbol{Q}}}$，这时有 $\phi(L_{\boldsymbol{P}}) \subseteq R_{\overline{\boldsymbol{Q}}}$.

若 $\overline{\boldsymbol{P}}_1$ 与 $\overline{\boldsymbol{P}}_2$ 线性相关，同理可得到对于某一固定的 $\boldsymbol{P} \in \mathrm{Pur}(H_m)$，存在 $\overline{\boldsymbol{P}} \in \mathrm{Pur}(H_m)$，使得 $\phi(L_{\boldsymbol{P}}) \subseteq L_{\overline{\boldsymbol{P}}}$.

现在有结论：对于固定的 $\boldsymbol{P}_0 \in \mathrm{Pur}(H_m)$，存在 $\overline{\boldsymbol{P}_0} \in \mathrm{Pur}(H_m)$，使得 $\phi(L_{\boldsymbol{P}_0}) \subseteq L_{\overline{\boldsymbol{P}_0}}$. 接着用这个结论证明：对 $\forall \boldsymbol{P} \in \mathrm{Pur}(H_m)$，存在 $\overline{\boldsymbol{P}} \in \mathrm{Pur}(H_m)$，使得 $\phi(L_{\boldsymbol{P}}) \subseteq L_{\overline{\boldsymbol{P}}}$.

记：

$$A = \{\boldsymbol{P} \in \mathrm{Pur}(H_m): \phi(L_{\boldsymbol{P}}) \in L_{\overline{\boldsymbol{P}}},\ \overline{\boldsymbol{P}} \in \mathrm{Pur}(H_m)\}$$

$$B = \{\boldsymbol{P} \in \mathrm{Pur}(H_m): \phi(L_{\boldsymbol{P}}) \in R_{\overline{\boldsymbol{Q}}},\ \overline{\boldsymbol{Q}} \in \mathrm{Pur}(H_n)\}$$

则 $A \cup B = \mathrm{Pur}(H_m)$. 假设 $B \neq \varnothing$，那么对 $\forall \boldsymbol{Q} \in \mathrm{Pur}(H_n)$，都存在 $\boldsymbol{P}_1 \in B$ 和 $\overline{\boldsymbol{Q}} \in \mathrm{Pur}(H_m)$，使得 $\phi(\boldsymbol{P}_1 \otimes \boldsymbol{Q}) = \overline{\boldsymbol{P}_{\boldsymbol{Q}}} \otimes \overline{\boldsymbol{Q}}$. 又因为 $\phi(L_{\boldsymbol{P}_0}) \subseteq L_{\overline{\boldsymbol{P}_0}}$，所以

$$\overline{\boldsymbol{P}}_{t_3} \otimes \overline{\boldsymbol{Q}}_{t_3} = \phi((t\boldsymbol{P}_1 + (1-t)\boldsymbol{P}_0) \otimes \boldsymbol{Q}) = \phi(t\boldsymbol{P}_1 \otimes \boldsymbol{Q} + (1-t)\boldsymbol{P}_0 \otimes \boldsymbol{Q}) =$$

$$t\phi(\boldsymbol{P}_1 \otimes \boldsymbol{Q}) + (1-t)\phi(\boldsymbol{P}_0 \otimes \boldsymbol{Q}) = t\,\overline{\boldsymbol{P}_{\boldsymbol{Q}}} \otimes \overline{\boldsymbol{Q}} + (1-t)\,\overline{\boldsymbol{P}_0} \otimes \overline{\boldsymbol{Q}_{\boldsymbol{P}_0}}$$

由于 $\overline{\boldsymbol{P}_{\boldsymbol{Q}}}$ 和 $\overline{\boldsymbol{Q}_{\boldsymbol{P}_0}}$ 的任意性，有 $\overline{\boldsymbol{P}_0} = \overline{\boldsymbol{P}_{\boldsymbol{Q}}}$ 和 $\overline{\boldsymbol{Q}} = \overline{\boldsymbol{Q}_{\boldsymbol{P}_0}}$. 于是

$$\phi(\boldsymbol{P}_1 \otimes \boldsymbol{Q}) = \overline{\boldsymbol{P}}_0 \otimes \overline{\boldsymbol{Q}} = \phi(\boldsymbol{P}_0 \otimes \boldsymbol{Q})$$

这与 ϕ 是单射矛盾. 所以 $B = \varnothing$，即对 $\forall \boldsymbol{P} \in \mathrm{Pur}(H_m)$，存在 $\overline{\boldsymbol{P}} \in \mathrm{Pur}(H_m)$，使得 $\phi(L_{\boldsymbol{P}}) \subseteq L_{\overline{\boldsymbol{P}}}$. 同理可证：当固定 $\boldsymbol{P}_0 \in \mathrm{Pur}(H_m)$ 时，存在 $\overline{\boldsymbol{Q}_0} \in \mathrm{Pur}(H_n)$，使得 $\phi(L_{\boldsymbol{P}_0}) \subseteq R_{\overline{\boldsymbol{Q}_0}}$ 时，对 $\forall \boldsymbol{P} \in \mathrm{Pur}(H_m)$，存在 $\overline{\boldsymbol{Q}} \in$

$\mathrm{Pur}(H_n)$，使得$\phi(L_{\boldsymbol{P}})\subseteq R_{\overline{\boldsymbol{Q}}}$ 成立.

引理 2　如果映射 ϕ：$S_{\mathrm{sep}}(H_m\otimes H_n)\longrightarrow S_{\mathrm{sep}}(H_m\otimes H_n)$ 所满足的条件与引理 1 的一样，那么下面结论之一成立：

(i) 对 $\forall \boldsymbol{Q}\in \mathrm{Pur}(H_n)$，至少存在一个$\overline{\boldsymbol{Q}}\in \mathrm{Pur}(H_n)$，使得 $\phi(R_{\boldsymbol{Q}})\subseteq R_{\overline{\boldsymbol{Q}}}$；

(ii) 对 $\forall \boldsymbol{Q}\in \mathrm{Pur}(H_n)$，至少存在一个$\overline{\boldsymbol{P}}\in \mathrm{Pur}(H_m)$，使得 $\phi(R_{\boldsymbol{Q}})\subseteq L_{\overline{\boldsymbol{P}}}$.

引理 3　如果映射 ϕ：$S_{\mathrm{sep}}(H_m\otimes H_n)\longrightarrow S_{\mathrm{sep}}(H_m\otimes H_n)$ 所满足的条件与引理 1 的一样，那么下面结论之一成立：

(i) 引理 1(i) 和引理 2(i) 都成立；

(ii) 引理 1(ii) 和引理 2(ii) 都成立.

证　如果引理 1(i) 和引理 2(ii) 同时成立，则对 $\forall \boldsymbol{P}\in \mathrm{Pur}(H_m)$ 和 $\boldsymbol{Q}\in \mathrm{Pur}(H_n)$，分别存在$\overline{\boldsymbol{P}},\overline{\overline{\boldsymbol{P}}}\in \mathrm{Pur}(H_m)$，使得 $\phi(L_{\boldsymbol{P}})\subseteq L_{\overline{\boldsymbol{P}}}$和$\phi(R_{\boldsymbol{Q}})\subseteq L_{\overline{\overline{\boldsymbol{P}}}}$ 同时成立，于是

$$\overline{\boldsymbol{P}}_{t_4}\otimes\overline{\boldsymbol{Q}}_{t_4}=\phi((t\boldsymbol{P}+(1-t)\boldsymbol{P})\otimes\boldsymbol{Q})=\phi(t\boldsymbol{P}\otimes\boldsymbol{Q}+(1-t)\boldsymbol{P}\otimes\boldsymbol{Q})=$$
$$t\phi(\boldsymbol{P}\otimes\boldsymbol{Q})+(1-t)\phi(\boldsymbol{P}\otimes\boldsymbol{Q})=t\,\overline{\boldsymbol{P}}\otimes\boldsymbol{Q}+(1-t)\,\overline{\overline{\boldsymbol{P}}}\otimes\boldsymbol{Q}$$

又因为 $\boldsymbol{Q}$ 是任意的，则有 $t\,\overline{\boldsymbol{P}}+(1-t)\,\overline{\boldsymbol{P}}=\boldsymbol{P}$. 注意到 $\boldsymbol{P}$ 是纯态，而且$\overline{\boldsymbol{P}},\overline{\overline{\boldsymbol{P}}}$ 是不同纯态，所以

$$t\,\overline{\boldsymbol{P}}+(1-t)\,\overline{\overline{\boldsymbol{P}}}\neq\boldsymbol{P}$$

矛盾. 所以引理 1(i) 和引理 2(ii) 不能同时成立. 同理，引理 1(ii) 和引理 2(i) 也不能同时成立.

定理 1 的证明　先假设引理 3(i) 成立，即对 $\forall \boldsymbol{P}\in \mathrm{Pur}(H_m)$，$\boldsymbol{Q}\in \mathrm{Pur}(H_n)$，有 $\phi(L_{\boldsymbol{P}})\subseteq L_{\overline{\boldsymbol{P}}}$，$\phi(R_{\boldsymbol{Q}})\subseteq R_{\overline{\boldsymbol{Q}}}$. 也就是对 $\forall \boldsymbol{P}\in \mathrm{Pur}(H_m)$，存在纯态 $\tau_1(\boldsymbol{P}),\tau_2(\boldsymbol{P},\boldsymbol{Q})$($\tau_2$ 与 $\boldsymbol{P}$ 有关)，使得等式 $\phi(\boldsymbol{P}\otimes\boldsymbol{Q})=\tau_1(\boldsymbol{P})\otimes\tau_2(\boldsymbol{P},\boldsymbol{Q})$ 对 $\forall \boldsymbol{Q}\in \mathrm{Pur}(H_n)$ 成立.

为了证明 $\tau_2(\boldsymbol{P},\boldsymbol{Q})$ 与 $\boldsymbol{P}$ 无关，现在假设 $\mathrm{Pur}(H_m)$ 中有两个不同的 $\boldsymbol{P}_1,\boldsymbol{P}_2$，于是：

$$\phi(\boldsymbol{P}_1\otimes\boldsymbol{Q})=\tau_1(\boldsymbol{P}_1)\otimes\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})\qquad\phi(\boldsymbol{P}_2\otimes\boldsymbol{Q})=\tau_1(\boldsymbol{P}_2)\otimes\tau_2(\boldsymbol{P}_2,\boldsymbol{Q})$$

则

$$\overline{\boldsymbol{P}}\otimes\overline{\boldsymbol{Q}}=\phi((t\boldsymbol{P}_1+(1-t)\boldsymbol{P}_2)\otimes\boldsymbol{Q})=\phi(t\boldsymbol{P}_1\otimes\boldsymbol{Q}+(1-t)\boldsymbol{P}_2\otimes\boldsymbol{Q})=$$
$$t\phi(\boldsymbol{P}_1\otimes\boldsymbol{Q})+(1-t)\phi(\boldsymbol{P}_2\otimes\boldsymbol{Q})=$$
$$t(\tau_1(\boldsymbol{P}_1)\otimes\tau_2(\boldsymbol{P}_1,\boldsymbol{Q}))+(1-t)(\tau_1(\boldsymbol{P}_2)\otimes\tau_2(\boldsymbol{P}_2,\boldsymbol{Q}))$$

这时，要么 $\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})$ 与 $\tau_2(\boldsymbol{P}_2,\boldsymbol{Q})$ 线性相关，要么 $\tau_1(\boldsymbol{P}_1)$ 与 $\tau_1(\boldsymbol{P}_2)$ 线性相关.

如果 $\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})$ 与 $\tau_2(\boldsymbol{P}_2,\boldsymbol{Q})$ 线性相关，则 $\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})=\tau_2(\boldsymbol{P}_2,\boldsymbol{Q})$，这时 τ_2 与 $\boldsymbol{P}$ 无关；如果 $\tau_1(\boldsymbol{P}_1)$ 与 $\tau_1(\boldsymbol{P}_2)$ 线性相关，则有

$$\phi((t\boldsymbol{P}_1+(1-t)\boldsymbol{P}_2)\otimes\boldsymbol{Q})=\tau_1(\boldsymbol{P}_1)\otimes(t\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})+(1-t)\tau_2(\boldsymbol{P}_2,\boldsymbol{Q}))$$

这时加上条件 $\phi(R_{\boldsymbol{Q}})\subseteq R_{\overline{\boldsymbol{Q}}}$，有

$$\phi((t\boldsymbol{P}_1+(1-t)\boldsymbol{P}_2)\otimes\boldsymbol{Q})=(t\tau_1(\boldsymbol{P}_1,\boldsymbol{Q})+(1-t)\tau_1(\boldsymbol{P}_2,\boldsymbol{Q}))\otimes\tau_2(\boldsymbol{Q})$$

于是 $t\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})+(1-t)\tau_2(\boldsymbol{P}_2,\boldsymbol{Q})=\tau_2(\boldsymbol{Q})$. 又因为 $\tau_2(\boldsymbol{P}_1,\boldsymbol{Q}),\tau_2(\boldsymbol{P}_2,\boldsymbol{Q}),\tau_2(\boldsymbol{Q})$ 全是纯态，所以

$$\tau_2(\boldsymbol{P}_1,\boldsymbol{Q})=\tau_2(\boldsymbol{P}_2,\boldsymbol{Q})=\tau_2(\boldsymbol{Q})$$

这样就证明了对于任意可分态 $\boldsymbol{P}\otimes\boldsymbol{Q}$，存在两个双射 τ_1：$\mathrm{Pur}(H_m)\longrightarrow\mathrm{Pur}(H_m)$，$\tau_2$：$\mathrm{Pur}(H_n)\longrightarrow\mathrm{Pur}(H_n)$（因为 ϕ 是双射），使得

$$\phi(\boldsymbol{P}\otimes\boldsymbol{Q})=\tau_1(\boldsymbol{P})\otimes\tau_2(\boldsymbol{Q})\tag{1}$$

当引理 3(ii) 成立时，同理可证对于任意可分态 $\boldsymbol{P}\otimes\boldsymbol{Q}$，存在两个双射$\bar{\tau}_1$：$\mathrm{Pur}(H_n)\longrightarrow\mathrm{Pur}(H_m)$，$\bar{\tau}_2$：$\mathrm{Pur}(H_m)\longrightarrow\mathrm{Pur}(H_n)$ 使得

$$\phi(\boldsymbol{P}\otimes\boldsymbol{Q})=\bar{\tau}_1(\boldsymbol{Q})\otimes\bar{\tau}_2(\boldsymbol{P})\tag{2}$$

接着断言：保 von Neumann 熵的映射 ϕ 是保持正交的，即对 $\forall \boldsymbol{\rho},\boldsymbol{\sigma}\in S_{\mathrm{sep}}(H_m\otimes H_n)$，若 $\boldsymbol{\rho\sigma}=\boldsymbol{0}$，则 $\phi(\boldsymbol{\rho})\phi(\boldsymbol{\sigma})=\boldsymbol{0}$. 事实上，在等式 $S(t\boldsymbol{\rho}+(1-t)\boldsymbol{\sigma})=S(t\phi(\boldsymbol{\rho})+(1-t)\phi(\boldsymbol{\sigma}))$ 中令 $t=1$，于是对 $\forall \boldsymbol{\rho}\in S_{\mathrm{sep}}(H_m\otimes H_n)$，有 $S(\boldsymbol{\rho})=S(\phi(\boldsymbol{\rho}))$. 由文献[8] 知

$$
\begin{aligned}
tS(\boldsymbol{\rho})+(1-t)S(\boldsymbol{\sigma})-\lambda_1\log\lambda_1-\lambda_2\log\lambda_2 &= S(t\boldsymbol{\rho}+(1-t)\boldsymbol{\sigma}) = \\
&S(t\phi(\boldsymbol{\rho})+(1-t)\phi(\boldsymbol{\sigma})) = \\
&tS(\phi(\boldsymbol{\rho}))+(1-t)S(\phi(\boldsymbol{\sigma}))-\lambda_1\log\lambda_1-\lambda_2\log\lambda_2
\end{aligned}
$$

也就是 $\phi(\boldsymbol{\rho})\phi(\boldsymbol{\sigma})=\mathbf{0}$，$\phi$ 是保正交的.

因为 $\dim(H_m\otimes H_n)<\infty$，于是 H_m 上的任意一个量子态 $\boldsymbol{\rho}$ 都可以写成 $\boldsymbol{\rho}=\sum_{i=1}^{m}\lambda_i\boldsymbol{P}_i$，其中 $\sum_{i=1}^{m}\lambda_i=1$，$\lambda_i\geqslant 0$，$\boldsymbol{P}_i\in \mathrm{Pur}(H_m)(i=1,\cdots,m)$. 在 $\mathrm{Pur}(H_n)$ 上取 $\boldsymbol{Q}$，注意到：

$$(\boldsymbol{P}_i\otimes\boldsymbol{Q})(\boldsymbol{P}_j\otimes\boldsymbol{Q})=\mathbf{0}\qquad i\neq j$$

$$S(\boldsymbol{P}_i\otimes\boldsymbol{Q})=S(\phi(\boldsymbol{P}_i\otimes\boldsymbol{Q}))$$

以及等式(1)，由文献[8]有

$$
\begin{aligned}
S(\phi(\boldsymbol{\rho}\otimes\boldsymbol{Q})) &= S(\boldsymbol{\rho}\otimes\boldsymbol{Q}) = S\Big(\sum_{i=1}^{m}\lambda_i\boldsymbol{P}_i\otimes\boldsymbol{Q}\Big)=\sum_{i=1}^{m}\lambda_i S(\boldsymbol{P}_i\otimes\boldsymbol{Q})-\sum_{i=1}^{m}\lambda_i\log\lambda_i = \\
&\sum_{i=1}^{m}\lambda_i S(\phi(\boldsymbol{P}_i\otimes\boldsymbol{Q}))-\sum_{i=1}^{m}\lambda_i\log\lambda_i = S\Big(\sum_{i=1}^{m}\lambda_i\phi(\boldsymbol{P}_i\otimes\boldsymbol{Q})\Big)=S\Big(\sum_{i=1}^{m}\lambda_i\tau_{1i}(\boldsymbol{P}_i)\otimes\tau_2(\boldsymbol{Q})\Big)
\end{aligned}
$$

于是由文献[9]知，存在酉矩阵或共轭酉矩阵 $\boldsymbol{W}$，使得

$$\phi(\boldsymbol{\rho}\otimes\boldsymbol{Q})=\boldsymbol{W}\Big(\sum_{i=1}^{m}\lambda_i\tau_{1i}(\boldsymbol{P}_i)\otimes\tau_2(\boldsymbol{Q})\Big)\boldsymbol{W}^*$$

不失一般性，令 $\boldsymbol{W}=\boldsymbol{I}_{mn}$，其中 $\boldsymbol{I}_{mn}$ 是 $H_m\otimes H_n$ 上的单位矩阵，则

$$\phi(\boldsymbol{\rho}\otimes\boldsymbol{Q})=\sum_{i=1}^{m}\lambda_i\tau_{1i}(\boldsymbol{P}_i)\otimes\tau_2(\boldsymbol{Q})$$

定义 $\tau_{1Q}(\boldsymbol{\rho})=\sum_{i=1}^{m}\lambda_i\tau_{1i}(\boldsymbol{P}_i)$，显然 $\tau_{1Q}(\boldsymbol{\rho})$ 是从 $S(H_m)$ 到 $S(H_m)$ 的映射，则

$$\phi(\boldsymbol{\rho}\otimes\boldsymbol{Q})=\tau_{1Q}(\boldsymbol{\rho})\otimes\tau_2(\boldsymbol{Q})\tag{3}$$

在等式(3)里，若 $\boldsymbol{\rho}=\frac{\boldsymbol{I}_m}{m}$($\boldsymbol{I}_m$ 是单位矩阵)，则 $\tau_{1Q}(\boldsymbol{\rho})=\frac{\boldsymbol{I}_m}{m}$. 事实上，由文献[5,8]，并注意到 $S(\phi(\boldsymbol{\rho}\otimes\boldsymbol{\sigma}))=S(\boldsymbol{\rho}\otimes\boldsymbol{\sigma})$ 且 $\tau_2(\boldsymbol{Q})$ 是纯态，得

$$S\left(\phi\left(\frac{\boldsymbol{I}_m}{m}\otimes\boldsymbol{Q}\right)\right)=S\left(\frac{\boldsymbol{I}_m}{m}\otimes\boldsymbol{Q}\right)=S\left(\frac{\boldsymbol{I}_m}{m}\right)=S\left(\tau_{1Q}\left(\frac{\boldsymbol{I}_m}{m}\right)\otimes\tau_2(\boldsymbol{Q})\right)=S\left(\tau_{1Q}\left(\frac{\boldsymbol{I}_m}{m}\right)\right)=\log n$$

在等式(3)里，映射 $\tau_{1Q}:S(H_m)\longrightarrow S(H_m)$ 满足等式

$$S\left(t\tau_{1Q}(\boldsymbol{\rho})+(1-t)\tau_{1Q}\left(\frac{\boldsymbol{I}_m}{m}\right)\right)=S\left(t\boldsymbol{\rho}+(1-t)\frac{\boldsymbol{I}_m}{m}\right)\tag{4}$$

事实上，

$$
\begin{aligned}
S\left(t\boldsymbol{\rho}+(1-t)\frac{\boldsymbol{I}_m}{m}\right) &= S\left(\left(t\boldsymbol{\rho}+(1-t)\frac{\boldsymbol{I}_m}{m}\right)\otimes\boldsymbol{Q}\right)=S\left(t\boldsymbol{\rho}\otimes\boldsymbol{Q}+(1-t)\frac{\boldsymbol{I}_m}{m}\otimes\boldsymbol{Q}\right)= \\
&S\left(t\phi(\boldsymbol{\rho}\otimes\boldsymbol{Q})+(1-t)\phi\left(\frac{\boldsymbol{I}_m}{m}\otimes\boldsymbol{Q}\right)\right)=S\left(t\tau_{1Q}(\boldsymbol{\rho})\otimes\tau_2(\boldsymbol{Q})+(1-t)\tau_{1Q}\left(\frac{\boldsymbol{I}_m}{m}\right)\otimes\tau_2(\boldsymbol{Q})\right)= \\
&S\left(\left(t\tau_{1Q}(\boldsymbol{\rho})+(1-t)\tau_{1Q}\left(\frac{\boldsymbol{I}_m}{m}\right)\right)\otimes\tau_2(\boldsymbol{Q})\right)=S\left(t\tau_{1Q}(\boldsymbol{\rho})+(1-t)\frac{\boldsymbol{I}_m}{m}\right)
\end{aligned}
$$

由等式(4)和文献[5]知，在 $S(H_m)$ 上存在酉矩阵或共轭酉矩阵 $\boldsymbol{U}_Q$，使得 $\tau_{1Q}(\boldsymbol{\rho})=\boldsymbol{U}_Q\boldsymbol{\rho}(\boldsymbol{U}_Q)^*$. 这时等式(3)就可以写成

$$\phi(\boldsymbol{\rho}\otimes\boldsymbol{Q})=(\boldsymbol{U}_Q\boldsymbol{\rho}(\boldsymbol{U}_Q)^*)\otimes\tau_2(\boldsymbol{Q})\tag{5}$$

等式(5)中的 $\boldsymbol{U}_Q$ 与 $\boldsymbol{Q}$ 无关. 事实上，在 $\mathrm{Pur}(H_n)$ 中取两个不同的量子态 $\boldsymbol{Q}_1,\boldsymbol{Q}_2$，使得对于 $\boldsymbol{\rho}\in S(H_m)$，有 $\tau_{1Q_i}(\boldsymbol{\rho})=\boldsymbol{U}_{Q_i}\boldsymbol{\rho}(\boldsymbol{U}_{Q_i})^*(i=1,2)$. 对于任意的 $\boldsymbol{P}\in\mathrm{Pur}(H_m)$，由等式(1),(3),(5)有：

$$\tau_1(\boldsymbol{P})\otimes\tau_2(\boldsymbol{Q}_1)=\phi(\boldsymbol{P}\otimes\boldsymbol{Q}_1)=\tau_{1Q_1}(\boldsymbol{P})\otimes\tau_2(\boldsymbol{Q}_1)=\boldsymbol{U}_{Q_1}\boldsymbol{P}(\boldsymbol{U}_{Q_1})^*\otimes\tau_2(\boldsymbol{Q}_1)$$

$$\tau_1(\boldsymbol{P})\otimes\tau_2(\boldsymbol{Q}_2)=\phi(\boldsymbol{P}\otimes\boldsymbol{Q}_2)=\tau_{1Q_2}(\boldsymbol{P})\otimes\tau_2(\boldsymbol{Q}_2)=\boldsymbol{U}_{Q_2}\boldsymbol{P}(\boldsymbol{U}_{Q_2})^*\otimes\tau_2(\boldsymbol{Q}_2)$$

又因为对 $\forall \boldsymbol{P} \in \mathrm{Pur}(H_m)$，有 $\phi(L_{\boldsymbol{P}}) \subseteq L_{\overline{\boldsymbol{P}}}$，于是 $\tau_1(\boldsymbol{P}) = \boldsymbol{U}_{Q_1}\boldsymbol{P}(\boldsymbol{U}_{Q_1})^* = \boldsymbol{U}_{Q_2}\boldsymbol{P}(\boldsymbol{U}_{Q_2})^*$. 再由文献[10]的引理 4.1 知 $\boldsymbol{U}_{Q_1} = \boldsymbol{U}_{Q_2}$. 若令 $\boldsymbol{U}_m = \boldsymbol{U}_{Q_1} = \boldsymbol{U}_{Q_2}$，则对 $\forall \boldsymbol{\rho} \in S(H_m)$ 和 $\boldsymbol{Q} \in \mathrm{Pur}(H_n)$，有

$$\phi(\boldsymbol{\rho} \otimes \boldsymbol{Q}) = (\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) \otimes \tau_2(\boldsymbol{Q}) \tag{6}$$

同理对 $\forall \boldsymbol{\sigma} \in S(H_n)$ 和 $\boldsymbol{P} \in \mathrm{Pur}(H_m)$，有

$$\phi(\boldsymbol{P} \otimes \boldsymbol{\sigma}) = \tau_1(\boldsymbol{P}) \otimes (\boldsymbol{V}_n\boldsymbol{\rho}(\boldsymbol{V}_n)^*) \tag{7}$$

等式(6)中的 $\tau_2(\boldsymbol{Q})$ 等于 $\boldsymbol{V}_n\boldsymbol{Q}(\boldsymbol{V}_n)^*$，其中 $\boldsymbol{V}_n$ 是 H_m 上的酉矩阵或共轭酉矩阵. 事实上，对 $\forall \boldsymbol{\rho} \in S(H_m)$，$\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathrm{Pur}(H_n)$，$t \in [0, 1]$，有

$$\begin{aligned}
S(\boldsymbol{\rho}) + S(t\boldsymbol{Q}_1 + (1-t)\boldsymbol{Q}_2) = & S(\boldsymbol{\rho} \otimes (t\boldsymbol{Q}_1 + (1-t)\boldsymbol{Q}_2)) = \\
& S(t\boldsymbol{\rho} \otimes \boldsymbol{Q}_1 + (1-t)\boldsymbol{\rho} \otimes \boldsymbol{Q}_2) = S(t\phi(\boldsymbol{\rho} \otimes \boldsymbol{Q}_1) + (1-t)\phi(\boldsymbol{\rho} \otimes \boldsymbol{Q}_2)) = \\
& S(t\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^* \otimes \tau_2(\boldsymbol{Q}_1) + (1-t)(\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) \otimes \tau_2(\boldsymbol{Q}_2)) = \\
& S(\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^* \otimes (t\tau_2(\boldsymbol{Q}_1) + (1-t)\tau_2(\boldsymbol{Q}_2))) = \\
& S(\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) + S(t\tau_2(\boldsymbol{Q}_1) + (1-t)\tau_2(\boldsymbol{Q}_2)) = \\
& S(\boldsymbol{\rho}) + S(t\tau_2(\boldsymbol{Q}_1) + (1-t)\tau_2(\boldsymbol{Q}_2))
\end{aligned}$$

于是

$$S(t\boldsymbol{Q}_1 + (1-t)\boldsymbol{Q}_2) = S(t\tau_2(\boldsymbol{Q}_1) + (1-t)\tau_2(\boldsymbol{Q}_2))$$

注意到映射 τ_2 是满射，所以由文献[5]知 $\tau_2(\boldsymbol{Q}) = \boldsymbol{V}_n\boldsymbol{Q}(\boldsymbol{V}_n)^*$. 于是等式(6)可以写成

$$\phi(\boldsymbol{\rho} \otimes \boldsymbol{Q}) = (\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) \otimes (\boldsymbol{V}_n\boldsymbol{Q}(\boldsymbol{V}_n)^*)$$

同理，等式(7)可以写成

$$\phi(\boldsymbol{P} \otimes \boldsymbol{\sigma}) = (\boldsymbol{U}_m\boldsymbol{P}(\boldsymbol{U}_m)^*) \otimes (\boldsymbol{V}_n\boldsymbol{\sigma}(\boldsymbol{V}_n)^*)$$

最后，对 $\forall \boldsymbol{\rho} \in S(H_m)$，$\boldsymbol{\sigma} \in S(H_n)$，令 $\boldsymbol{\sigma} = \sum_{j=1}^{l} \mu_j \boldsymbol{Q}_j\,(l = 1, \cdots, n)$. 有

$$\begin{aligned}
S(\phi(\boldsymbol{\rho} \otimes \boldsymbol{\sigma})) = & S^r(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) = S\Big(\boldsymbol{\rho} \otimes \sum_{j=1}^{l}\mu_j\boldsymbol{Q}_j\Big) = \sum_{j=1}^{l}\mu_j S(\boldsymbol{\rho} \otimes \boldsymbol{Q}_j) - \sum_{j=1}^{l}\mu_j \log \mu_j = \\
& \sum_{j=1}^{l}\mu_j S(\phi(\boldsymbol{\rho} \otimes \boldsymbol{Q}_j)) - \sum_{j=1}^{l}\mu_j \log \mu_j = S\Big(\sum_{j=1}^{l}\mu_j\phi(\boldsymbol{\rho} \otimes \boldsymbol{Q}_j)\Big) = \\
& S\Big(\sum_{j=1}^{l}\mu_j(\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) \otimes (\boldsymbol{V}_n\boldsymbol{Q}_j(\boldsymbol{V}_n)^*)\Big) = S((\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) \otimes (\boldsymbol{V}_n\boldsymbol{\sigma}(\boldsymbol{V}_n)^*))
\end{aligned}$$

这时在 H_m, H_n 上分别存在酉矩阵或共轭酉矩阵 $\boldsymbol{W}_m, \boldsymbol{W}_n$，使得

$$\begin{aligned}
\phi(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) = & (\boldsymbol{W}_m \otimes \boldsymbol{W}_n)((\boldsymbol{U}_m\boldsymbol{\rho}(\boldsymbol{U}_m)^*) \otimes (\boldsymbol{V}_n\boldsymbol{\sigma}(\boldsymbol{V}_n)^*))(\boldsymbol{W}_m \otimes \boldsymbol{W}_n)^* = \\
& ((\boldsymbol{W}_m\boldsymbol{U}_m) \otimes (\boldsymbol{W}_n\boldsymbol{V}_n))(\boldsymbol{\rho} \otimes \boldsymbol{\sigma})((\boldsymbol{W}_m\boldsymbol{U}_m) \otimes (\boldsymbol{W}_n\boldsymbol{V}_n))^*
\end{aligned}$$

令 $\overline{\boldsymbol{U}}_m = \boldsymbol{W}_m\boldsymbol{U}_m$，$\overline{\boldsymbol{V}}_n = \boldsymbol{W}_n\boldsymbol{V}_n$，那么对 $\forall \boldsymbol{\rho} \in S(H_m)$，$\boldsymbol{\sigma} \in S(H_n)$，有

$$\phi(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) = (\overline{\boldsymbol{U}}_m \otimes \overline{\boldsymbol{V}}_n)(\boldsymbol{\rho} \otimes \boldsymbol{\sigma})(\overline{\boldsymbol{U}}_m \otimes \overline{\boldsymbol{V}}_n)^*$$

若等式(2)成立，则有 $\phi(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) = (\overline{\boldsymbol{V}}_n \otimes \overline{\boldsymbol{U}}_m)(\boldsymbol{\sigma} \otimes \boldsymbol{\rho})(\overline{\boldsymbol{V}}_n \otimes \overline{\boldsymbol{U}}_m)^*$.

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Characterization of Quantum Channels Preserving von Neumann Entropy in Bipartite Systems

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Abstract: Let H_m be the complex Hilbert space with dim $H=m$, $S(H_m\otimes H_n)$ be all the quantum states acting on complex bipartite Hilbert space $H_m\otimes H_n$ and $S_{\text{sep}}(H_m\otimes H_n)$ be the convex set of comparable quantum states, ϕ: $S(H_m\otimes H_n)\longrightarrow S(H_m\otimes H_n)$ be quantum channels and $\phi(S_{\text{sep}}(H_m\otimes H_n))=S_{\text{sep}}(H_m\otimes H_n)$. Then ϕ satisfies von Neumann entropy $S(t\boldsymbol{\rho}+(1-t)\boldsymbol{\sigma})=S(t\phi(\boldsymbol{\rho})+(1-t)\phi(\boldsymbol{\sigma}))$ ($\forall t\in[0,1]$, $\forall \boldsymbol{\rho},\boldsymbol{\sigma}\in S_{\text{sep}}(H_m\otimes H_n)$) if and only if there exist unitary operators $\overline{\boldsymbol{U}}_m, \overline{\boldsymbol{V}}_n$ acting on H_m, H_n respectively such that $\phi(\boldsymbol{\rho})=(\overline{\boldsymbol{U}}_m\otimes\overline{\boldsymbol{V}}_n)\boldsymbol{\rho}(\overline{\boldsymbol{U}}_m\otimes\overline{\boldsymbol{V}}_n)^*$, $\forall \boldsymbol{\rho}\in S_{\text{sep}}(H_m\otimes H_n)$.

Key words: quantum channels; von Neumann entropy; quantum states

责任编辑 廖 坤