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# 一类含有未知导函数的三重积分不等式中未知函数的估计[①]

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**摘要**：研究了一类非线性三重积分不等式，其中被积函数中含有未知函数及其导函数，积分项外包含了非常数项. 利用变量替换技巧、放大技巧和反函数技巧等分析手段，给出了三重积分-微分不等式中未知函数的显上界估计，推广了已有结果. 最后举例说明所得结果可以用来研究微分一积分方程解的估计.

**关 键 词**：非线性积分不等式；含有未知导函数的三重积分；微分-积分方程；显式估计

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文献[1-2] 在研究微分方程的解对参数的连续依赖性时，建立了下面的积分不等式

$$u(t) \leqslant c + \int_a^t f(s)u(s)\mathrm{d}s \qquad t \in [a, b]$$

其中 $c \geqslant 0$ 是常数，给出了不等式中未知函数的估计

$$u(t) \leqslant c\exp\left(\int_a^t f(s)\mathrm{d}s\right) \qquad t \in [a, b] \tag{1}$$

大部分研究者研究积分号内不含未知函数的导函数的积分不等式[3-12]. 由于积分号内包含未知函数及其导函数的积分不等式在研究微分-积分方程中具有重要作用，文献[13] 定理 1.7.3,1.8.1,1.8.2 中研究了下面的积分号内含有未知函数及其导函数的线性积分不等式

$$\dot{u}(t) \leqslant a(t) + b(t)\int_0^t c(s)(u(s) + \dot{u}(s))\mathrm{d}s,\ t \in \mathbb{R}_+ \tag{2}$$

$$\dot{u}(t) \leqslant u(0) + \int_0^t a(s)(u(s) + \dot{u}(s))\mathrm{d}s + \int_0^t a(s)\left(\int_0^s b(\sigma)\dot{u}(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s,\ t \in \mathbb{R}_+ \tag{3}$$

$$\begin{aligned} u(t) \leqslant & k(t) + p(t)\left\{\int_0^t f(s)u(s)\mathrm{d}s + \int_0^t f(s)p(s)\left(\int_0^s g(\tau)u(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \right. \\ & \left.\int_0^t f(s)p(s)\left[\int_0^s g(\tau)p(\tau)\left(\int_0^\tau h(\sigma)u(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right]\mathrm{d}s\right\} \end{aligned} \tag{4}$$

文献[14] 进一步研究了时标上的线性积分不等式

$$u^\Delta(t) \leqslant a(t) + b(t)\int_{t_0}^t c(s)(u(s) + u^\Delta(s))\Delta s,\ t \in \mathbb{T}_0 \tag{5}$$

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$$u(t) \leqslant w(t) + p(t)\int_{t_0}^{t} \left\{[a(\tau) + b(\tau)]u(\tau) + b(\tau)p(\tau)\int_{t_0}^{\tau}[c(s)u(s) + d(s)]\Delta s\right\}\Delta\tau,\ t \in \mathbb{T}_0 \tag{6}$$

$$u(t) \leqslant u_0 + \int_{t_0}^{t} b(s)\left\{u(s) + \int_{t_0}^{s} c(\tau)\left[u(\tau) + \int_{t_0}^{\tau} q(\gamma)u(\gamma)\Delta\gamma\right]\Delta\tau\right\}\Delta s,\ t \in \mathbb{T}_0 \tag{7}$$

文献[15] 研究了积分号内含有未知函数及其导函数的非线性积分不等式

$$\dot{u}(t) \leqslant c + \int_{0}^{t} k(s)\dot{u}(s)(\dot{u}^{p}(s) + u^{2}(s))\mathrm{d}s,\ t \in \mathbb{R}_+ \tag{8}$$

其中：$c$ 是正常数，$p \geqslant 1$.

受文献[13-15] 的启发，本文研究了积分号外具有非常数因子，且积分号内含有未知函数及其导函数的非线性三重积分不等式

$$\begin{aligned}\dot{u}(t) \leqslant & q(t) + p(t)\left\{u(t) + \int_{t_0}^{t} f(s)(u(s) + \dot{u}(s))\mathrm{d}s + \right.\\ & \int_{t_0}^{t} f(s)p(s)\left(\int_{t_0}^{s} g(\tau)(u(\tau) + \dot{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \int_{t_0}^{t} f(s)p(s) \times \\ & \left.\left[\int_{t_0}^{s} g(\tau)p(\tau)\left(\int_{t_0}^{\tau} h(\sigma)\dot{u}(\sigma)(u^{2}(\sigma) + \dot{u}^{\theta}(\sigma)\ \mathrm{d}\sigma\right)\mathrm{d}\tau\right]\mathrm{d}s\right\},\ t \in [t_0,\ \infty)\end{aligned} \tag{9}$$

不等式(9) 把文献[13] 中的不等式(3) 推广成非线性积分不等式，把文献[15] 中的不等式(8) 推广成积分号外具有非常数因子的三重积分不等式. 本文利用分析技巧给出了不等式(9) 中未知函数的估计. 最后举例说明了本文研究结果可用来研究相应类型的微分-积分方程解的估计.

## 1　主要结果与证明

为了使结果的证明过程简单明了，先给出下面的引理.

**引理 1**[16-17]　令 $y \geqslant 0$，$p \geqslant q \geqslant 0$ 和 $p \neq 0$，则对任意 $K > 0$ 有关系式

$$y^{\frac{q}{p}} \leqslant \frac{q}{p}K^{\frac{q-p}{p}}y + \frac{p-q}{p}K^{\frac{q}{p}} \tag{10}$$

**引理 2**　假设函数 $u(t)$,$a(t)$,$b(t)$,$c(t)$,$d(t)$ 都是定义在$[t_0,\ \infty)$ 上的非负连续函数，且满足不等式

$$\begin{aligned}u(t) \leqslant & u(0) + \int_{t_0}^{t} a(s)\mathrm{d}s + \int_{t_0}^{t} b(s)u(s)\mathrm{d}s + \int_{t_0}^{t} c(s)u^{2}(s)\mathrm{d}s + \\ & \int_{t_0}^{t} d(s)u^{3}(s)\mathrm{d}s,\ t \in [t_0,\ \infty)\end{aligned} \tag{11}$$

如果 $u(0) > 0$，$(\exp(-(\ln(u(0) + \int_{t_0}^{t} a(s)\mathrm{d}s) - \int_{t_0}^{t} b(s)\mathrm{d}s)) - \int_{t_0}^{t} c(s)\mathrm{d}s)^{2} - \int_{t_0}^{t} 2\mathrm{d}(s)\mathrm{d}s > 0$，则有未知函数 $u(t)$ 的估计式

$$\begin{aligned}u(t) \leqslant & \left\{\left[\exp\left(-\ln\left(u(0) + \int_{t_0}^{t} a(s)\mathrm{d}s\right) - \int_{t_0}^{t} b(s)\mathrm{d}s\right) - \int_{0}^{t} c(s)\mathrm{d}s\right]^{2} - \right.\\ & \left.\int_{t_0}^{t} 2\mathrm{d}(s)\mathrm{d}s\right\}^{-\frac{1}{2}},\ t \in [t_0,\ \infty)\end{aligned} \tag{12}$$

**证**　对于任意非负实数 $T \in [t_0,\ \infty)$，由(11) 式可以看出

$$\begin{aligned}u(t) \leqslant & u(0) + \int_{t_0}^{T} a(s)\mathrm{d}s + \int_{t_0}^{t} b(s)u(s)\mathrm{d}s + \int_{t_0}^{t} c(s)u^{2}(s)\mathrm{d}s + \\ & \int_{t_0}^{t} \mathrm{d}(s)u^{3}(s)\mathrm{d}s,\ t \in [t_0,\ T]\end{aligned} \tag{13}$$

由(13) 式右端定义函数 $v(t)$，即

$$\begin{aligned}v(t) = & u(0) + \int_{t_0}^{T} a(s)\mathrm{d}s + \int_{t_0}^{t} b(s)u(s)\mathrm{d}s + \int_{t_0}^{t} c(s)u^{2}(s)\mathrm{d}s + \\ & \int_{t_0}^{t} d(s)u^{3}(s)\mathrm{d}s,\ t \in [t_0,\ T]\end{aligned} \tag{14}$$

由定义式(14)可看出

$$u(t) \leqslant v(t),\ v(t_0) = u(0) + \int_{t_0}^{T} a(s)\mathrm{d}s,\ t \in [t_0,\ T] \tag{15}$$

再求函数 $v(t)$ 的导函数，利用(15)式得

$$\dot{v}(t) = b(t)u(t) + c(t)u^2(t) + d(t)u^3(t) \leqslant b(t)v(t) + c(t)v^2(t) + d(t)v^3(t),\ t \in [t_0,\ T] \tag{16}$$

把不等式(16)两边同时除以 $v(t)$ 得到

$$\frac{\dot{v}(t)}{v(t)} \leqslant b(t) + c(t)v(t) + d(t)v^2(t),\ t \in [t_0,\ T] \tag{17}$$

先把(17)式中的 $t$ 替换成 $s$，然后两边关于 $s$ 从 $t_0$ 到 $t$ 积分，得到

$$\begin{aligned} \ln v(t) \leqslant & \ln v(t_0) + \int_{t_0}^{t} b(s)\mathrm{d}s + \int_{t_0}^{t} c(s)v(s)\mathrm{d}s + \int_{t_0}^{t} d(s)v^2(s)\mathrm{d}s \leqslant \\ & \ln v(t_0) + \int_{t_0}^{T} b(s)\mathrm{d}s + \int_{t_0}^{t} c(s)v(s)\mathrm{d}s + \int_{t_0}^{t} d(s)v^2(s)\mathrm{d}s,\ t \in [t_0,\ T] \end{aligned} \tag{18}$$

将不等式(18)的右端定义为函数 $w(t)$，即

$$w(t) = \ln v(t_0) + \int_{t_0}^{T} b(s)\mathrm{d}s + \int_{t_0}^{t} c(s)v(s)\mathrm{d}s + \int_{t_0}^{t} d(s)v^2(s)\mathrm{d}s,\ t \in [t_0,\ T] \tag{19}$$

由(18)式和(19)式可以看出 $w(t)$ 是非负连续增函数，且满足

$$w(t_0) = \ln v(t_0) + \int_{t_0}^{T} b(s)\mathrm{d}s,\ v(t) \leqslant \exp(w(t)),\ t \in [t_0,\ T] \tag{20}$$

求函数 $w(t)$ 的导函数得到

$$\dot{w}(t) = c(t)v(t) + d(t)v^2(t) \leqslant c(t)\exp(w(t)) + d(t)\exp(2w(t)),\ t \in [t_0,\ T] \tag{21}$$

不等式(21)两边同除以 $-\exp(w(t))$ 得到

$$-\exp(-w(t))\,\dot{w}(t) \geqslant -c(t) - d(t)\exp(w(t)),\ t \in [t_0,\ T] \tag{22}$$

然后把不等式(22)中的 $t$ 改写成 $s$，两边再关于 $s$ 从 0 到 $t$ 积分，得

$$\begin{aligned} \exp(-w(t)) \geqslant & \exp(-w(t_0)) - \int_{t_0}^{t} c(s)\mathrm{d}s - \int_{t_0}^{t} d(s)\exp(w(s))\mathrm{d}s \geqslant \\ & \exp(-w(t_0)) - \int_{t_0}^{T} c(s)\mathrm{d}s - \int_{t_0}^{t} d(s)\exp(w(s))\mathrm{d}s\ t \in [t_0,\ T] \end{aligned} \tag{23}$$

将不等式(23)的右端定义为函数 $z(t)$，即

$$z(t) = \exp(-w(t_0)) - \int_{t_0}^{T} c(s)\mathrm{d}s - \int_{t_0}^{t} d(s)\exp(w(s))\mathrm{d}s\ t \in [t_0,\ T] \tag{24}$$

由(23)式和(24)式可以看出 $z(t)$ 是连续减函数，且满足

$$z(t_0) = \exp(-w(t_0)) - \int_{t_0}^{T} c(s)\mathrm{d}s,\ z(t) \leqslant \exp(-w(t)),\ t \in [t_0,\ T] \tag{25}$$

求函数 $z(t)$ 的导函数得到

$$\dot{z}(t) = -d(t)\exp(w(t)) \geqslant -\frac{d(t)}{z(t)},\ t \in [t_0,\ T] \tag{26}$$

不等式(26)两边同乘 $z(t)$ 得到

$$z(t)\,\dot{z}(t) \geqslant -d(t),\ t \in [t_0,\ T] \tag{27}$$

然后把不等式(27)中的 $t$ 改写成 $s$，两边再关于 $s$ 从 0 到 $t$ 积分，得

$$z^2(t) \geqslant z^2(t_0) - \int_{t_0}^{t} 2d(s)\mathrm{d}s,\ t \in [t_0,\ T] \tag{28}$$

综合(15),(20),(25)和(28)式推出

$$u(t) \leqslant v(t) \leqslant \exp(w(t)) \leqslant \frac{1}{z(t)} \leqslant \left(z^2(t_0) - \int_{t_0}^{t} 2d(s)\mathrm{d}s\right)^{-\frac{1}{2}} =$$

$$\left[\left(\exp(-w(t_0))-\int_0^T c(s)\mathrm{d}s\right)^2-\int_{t_0}^t 2d(s)\mathrm{d}s\right]^{-\frac{1}{2}}=$$

$$\left\{\left[\exp-\ln v(t_0)-\int_{t_0}^T b(s)\mathrm{d}s-\int_0^T c(s)\mathrm{d}s\right]^2-\int_{t_0}^t 2d(s)\mathrm{d}s\right\}^{-\frac{1}{2}}=$$

$$\left\{\left[\exp-\ln\left(u(0)+\int_{t_0}^T a(s)\mathrm{d}s\right)-\int_{t_0}^T b(s)\mathrm{d}s-\int_0^T c(s)\mathrm{d}s\right]^2-\int_{t_0}^t 2d(s)\mathrm{d}s\right\}^{-\frac{1}{2}},\ t\in[0,T] \tag{29}$$

在(29) 式中令 $t=T$，得到

$$u(T)\leqslant\left\{\left[\exp\left(-\ln\left(u(0)+\int_{t_0}^T a(s)\mathrm{d}s\right)-\int_{t_0}^T b(s)\mathrm{d}s\right)-\int_0^T c(s)\mathrm{d}s\right]^2-\int_{t_0}^T 2d(s)\mathrm{d}s\right\}^{-\frac{1}{2}} \tag{30}$$

因为 $T$ 是任意的，可以把(30) 式写成

$$u(t)\leqslant\left\{\left[\exp\left(-\ln\left(u(0)+\int_{t_0}^t a(s)\mathrm{d}s\right)-\int_{t_0}^t b(s)\mathrm{d}s\right)-\int_0^t c(s)\mathrm{d}s\right]^2-\int_{t_0}^t 2d(s)\mathrm{d}s\right\}^{-\frac{1}{2}},\ t\in[t_0,\infty) \tag{31}$$

这正是所要证明的估计式(12).

**定理 1**　假设 $q(t),p(t),f(t),g(t),h(t)$ 都是定义在$[t_0,\infty)$ 上的非负连续已知函数，$\theta=\dfrac{\theta_2}{\theta_1}<1$ 是正常数，$u(t)$ 和 $\dot{u}(t)$ 是定义在$[t_0,\infty)$ 上的满足不等式(9) 的非负未知函数，$u(t_0)>0$. 对于任意 $t\in[t_0,\infty)$，如果

$$\left(\exp\left(-\left(\ln\left(u(t_0)+\int_{t_0}^t A(s)\mathrm{d}s\right)-\int_{t_0}^t B(s)\mathrm{d}s\right)\right)-\int_{t_0}^t C(s)\mathrm{d}s\right)^2-\int_{t_0}^t 2D(s)\mathrm{d}s>0 \tag{32}$$

则对于任意 $K>0$，有未知函数 $u(t)$ 的估计式

$$u(t)\leqslant u(t_0)+\int_{t_0}^t (q(s)+p(s)Z(s))\mathrm{d}s,\ t\in[t_0,\infty) \tag{33}$$

其中

$$Z(t):=u(t_0)\exp\left(\int_{t_0}^t (p(s)+f(s))\mathrm{d}s\right)+\int_{t_0}^t [q(s)(1+f(s))+f(s)p(s)R(s)]\exp\left(\int_s^t (p(\tau)+f(\tau))\mathrm{d}\tau\right)\mathrm{d}s \tag{34}$$

$$R(t):=u(t_0)+\int_{t_0}^t \{q(s)(1+f(s)+g(s))+[p(s)(1+f(s)+g(s))+f(s)+g(s)]M(s)\}\mathrm{d}s \tag{35}$$

$$M(t):=\left(\left(\exp\left(-\left(\ln\left(u(t_0)+\int_{t_0}^t A(s)\mathrm{d}s\right)-\int_{t_0}^t B(s)\mathrm{d}s\right)\right)-\int_{t_0}^t C(s)\mathrm{d}s\right)^2-\int_{t_0}^t 2D(s)\mathrm{d}s\right)^{-\frac{1}{2}} \tag{36}$$

$$A(t):=q(t)\left(1+f(t)+g(t)+\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}q(t)h(t)+\frac{\theta_2-\theta_1}{\theta_2}K^{\frac{\theta_1}{\theta_2}}h(t)\right) \tag{37}$$

$$B(t):=p(t)+f(t)+g(t)+f(t)p(t)+g(t)p(t)+h(t)q(t)\left(\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}\right)+h(t)p(t)\left(\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}q(t)+\frac{\theta_2-\theta_1}{\theta_2}K^{\frac{\theta_1}{\theta_2}}\right) \tag{38}$$

$$C(t):=h(t)p(t)\left(\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}\right)+h(t)q(t) \tag{39}$$

$$D(t):=h(t)p(t) \tag{40}$$

**证** 由不等式(9)定义函数 $z(t)$

$$z(t):=u(t)+\int_{t_0}^{t}f(s)(u(s)+\dot{u}(s))\mathrm{d}s+\int_{t_0}^{t}f(s)p(s)\left(\int_{t_0}^{s}g(\tau)(u(\tau)+\dot{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\int_{t_0}^{t}f(s)p(s)\times\left[\int_{t_0}^{s}g(\tau)p(\tau)\left(\int_{t_0}^{\tau}h(\sigma)\dot{u}(\sigma)(u^2(\sigma)+\dot{u}^{\theta}(\sigma))\mathrm{d}\sigma\right)\mathrm{d}\tau\right]\mathrm{d}s,\ t\in[t_0,\infty) \tag{41}$$

由(9)式和(41)式可看出 $z(t)$ 是非减函数，且有

$$z(t_0)=u(t_0),\ u(t)\leqslant z(t),\ \dot{u}(t)\leqslant q(t)+p(t)z(t),\ t\in[t_0,\infty) \tag{42}$$

求(41)式定义的函数 $z(t)$ 的导函数

$$\dot{z}(t)=\dot{u}(t)+f(t)(u(t)+\dot{u}(t))+f(t)p(t)\left[\int_{t_0}^{t}g(\tau)(u(\tau)+\dot{u}(\tau))\mathrm{d}\tau\right]+f(t)p(t)\left[\int_{t_0}^{t}g(\tau)p(\tau)\left(\int_{t_0}^{\tau}h(\sigma)\dot{u}(\sigma)(u^2(\sigma)+\dot{u}^{\theta}(\sigma))\mathrm{d}\sigma\right)\mathrm{d}\tau\right],\ t\in[t_0,\infty) \tag{43}$$

把(42)式代入(43)式得到

$$\begin{aligned}\dot{z}(t)\leqslant\ & q(t)+p(t)z(t)+f(t)(z(t)+q(t)+p(t)z(t))+f(t)p(t)\int_{t_0}^{t}g(\tau)(z(\tau)+\\ & q(\tau)+p(\tau)z(\tau))\mathrm{d}\tau+f(t)p(t)\int_{t_0}^{t}g(\tau)p(\tau)\left(\int_{t_0}^{\tau}h(\sigma)(q(\sigma)+\right.\\ & \left.p(\sigma)z(\sigma))(z^2(\sigma)+(q(\sigma)+p(\sigma)z(\sigma))^{\theta})\mathrm{d}\sigma\right)\mathrm{d}\tau\leqslant\\ & q(t)(1+f(t))+(p(t)+f(t))z(t)+f(t)p(t)\left\{z(t)+\int_{t_0}^{t}g(\tau)(z(\tau)+\right.\\ & q(\tau)+p(\tau)z(\tau))\mathrm{d}\tau+\int_{t_0}^{t}g(\tau)p(\tau)\left[\int_{t_0}^{\tau}h(\sigma)\big(q(\sigma)+\right.\\ & \left.\left.p(\sigma)z(\sigma))(z^2(\sigma)+(q(\sigma)+p(\sigma)z(\sigma))^{\theta}\big)\mathrm{d}\sigma\right]\mathrm{d}\tau\right\},\ t\in[t_0,\infty)\end{aligned} \tag{44}$$

再定义函数 $r(t)$

$$r(t):=z(t)+\int_{t_0}^{t}g(\tau)(z(\tau)+q(\tau)+p(\tau)z(\tau))\mathrm{d}\tau+\int_{t_0}^{t}g(\tau)p(\tau)\left(\int_{t_0}^{\tau}h(\sigma)(q(\sigma)+p(\sigma)z(\sigma))(z^2(\sigma)+(q(\sigma)+p(\sigma)z(\sigma))^{\theta})\,\mathrm{d}\sigma\right)\mathrm{d}\tau,\ t\in[t_0,\infty) \tag{45}$$

从定义式(45)可以看出 $r(t)$ 是非减函数，且有

$$r(t_0)=z(t_0),\ z(t)\leqslant r(t),\ t\in[t_0,\infty) \tag{46}$$

把(45)式和(46)式代入(44)式得

$$\dot{z}(t)\leqslant q(t)(1+f(t))+[p(t)+f(t)+f(t)p(t)]r(t),\ t\in[t_0,\infty) \tag{47}$$

再求函数 $r(t)$ 的导函数得

$$\dot{r}(t)=\dot{z}(t)+g(t)(z(t)+q(t)+p(t)z(t))+g(t)p(t)\int_{t_0}^{t}h(\sigma)(q(\sigma)+p(\sigma)z(\sigma))(z^2(\sigma)+(q(\sigma)+p(\sigma)z(\sigma))^{\theta})\mathrm{d}\sigma,\ t\in[t_0,\infty) \tag{48}$$

再把(46)式和(47)式代入(48)式得

$$\begin{aligned}\dot{r}(t)\leqslant\ & q(t)(1+f(t))+[p(t)+f(t)+f(t)p(t)]r(t)+g(t)(r(t)+q(t)+p(t)r(t))+\\ & g(t)p(t)\int_{t_0}^{t}h(\sigma)(q(\sigma)+p(\sigma)r(\sigma))(r^2(\sigma)+(q(\sigma)+p(\sigma)r(\sigma))^{\theta})\mathrm{d}\sigma=\end{aligned}$$

$$q(t)(1+f(t)+g(t))+[p(t)+f(t)+g(t)+f(t)p(t)]r(t)+g(t)p(t)\{r(t)+\int_{t_0}^{t}h(\sigma)(q(\sigma)+p(\sigma)r(\sigma))(r^2(\sigma)+(q(\sigma)+p(\sigma)r(\sigma))^{\theta})\mathrm{d}\sigma\},\ t\in[t_0,\infty) \tag{49}$$

为了进一步简化，再定义函数 $m(t)$

$$m(t)=r(t)+\int_{t_0}^{t}h(\sigma)(q(\sigma)+p(\sigma)r(\sigma))(r^2(\sigma)+(q(\sigma)+p(\sigma)r(\sigma))^{\theta})\mathrm{d}\sigma,\ t\in[t_0,\infty) \tag{50}$$

从定义式(50) 可以看出 $m(t)$ 是非减函数，且

$$m(t_0)=r(t_0),\ r(t)\leqslant m(t),\ t\in[t_0,\infty) \tag{51}$$

求函数 $m(t)$ 的导函数，利用(49),(50) 和(51) 式得

$$\begin{aligned}\dot m(t)=&\dot r(t)+h(t)(q(t)+p(t)r(t))(r^2(t)+(q(t)+p(t)r(t))^{\theta})\leqslant\\&q(t)(1+f(t)+g(t))+[p(t)+f(t)+g(t)+f(t)p(t)]r(t)+g(t)p(t)m(t)+\\&h(t)(q(t)+p(t)r(t))(r^2(t)+(q(t)+p(t)r(t))^{\theta})\leqslant\\&q(t)(1+f(t)+g(t))+[p(t)+f(t)+g(t)+f(t)p(t)]m(t)+g(t)p(t)m(t)+\\&h(t)(q(t)+p(t)m(t))(m^2(t)+(q(t)+p(t)m(t))^{\theta}),\ t\in[t_0,\infty)\end{aligned} \tag{52}$$

对于任意 $K>0$，利用引理 1 可以推出

$$\begin{aligned}(q(t)+p(t)m(t))^{\theta}\leqslant&\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}(q(t)+p(t)m(t))+\frac{\theta_2-\theta_1}{\theta_2}K^{\frac{\theta_1}{\theta_2}}\leqslant\\&\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}q(t)+\frac{\theta_2-\theta_1}{\theta_2}K^{\frac{\theta_1}{\theta_2}}+\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}p(t)m(t)\end{aligned} \tag{53}$$

把(53) 式代入(52) 式可得

$$\begin{aligned}\dot m(t)\leqslant&q(t)\left(1+f(t)+g(t)+\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}q(t)h(t)+\frac{\theta_2-\theta_1}{\theta_2}K^{\frac{\theta_1}{\theta_2}}h(t)\right)+\\&\left[p(t)+f(t)+g(t)+f(t)p(t)+g(t)p(t)+h(t)q(t)\left(\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}\right)+\right.\\&\left.h(t)p(t)\left(\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}q(t)+\frac{\theta_2-\theta_1}{\theta_2}K^{\frac{\theta_1}{\theta_2}}\right)\right]m(t)+\\&\left(h(t)p(t)\left(\frac{\theta_1}{\theta_2}K^{\frac{\theta_1-\theta_2}{\theta_2}}\right)+h(t)q(t)\right)m^2(t)+h(t)p(t)m^3(t)=\\&A(t)+B(t)m(t)+C(t)m^2(t)+D(t)m^3(t),\ t\in[t_0,\infty)\end{aligned} \tag{54}$$

其中：$A(t)$,$B(t)$,$C(t)$,$D(t)$ 由定理 1 中的(37),(38),(39) 和(40) 式定义. 先把不等式(54) 中的 $t$ 改写成 $s$，然后对不等式两边关于 $s$ 从 $t_0$ 到 $t$ 积分，得到

$$\begin{aligned}m(t)\leqslant&m(t_0)+\int_{t_0}^{t}A(s)\mathrm{d}s+\int_{t_0}^{t}B(s)m(s)\mathrm{d}s+\int_{t_0}^{t}C(s)m^2(s)\mathrm{d}s+\\&\int_{t_0}^{t}D(s)m^3(s)\mathrm{d}s,\ t\in\mathbb{R}_+\end{aligned} \tag{55}$$

利用(42),(46),(51) 式将(55) 式改写成

$$\begin{aligned}m(t)\leqslant&u(t_0)+\int_{t_0}^{t}A(s)\mathrm{d}s+\int_{t_0}^{t}B(s)m(s)\mathrm{d}s+\int_{t_0}^{t}C(s)m^2(s)\mathrm{d}s+\\&\int_{t_0}^{t}D(s)m^3(s)\mathrm{d}s,\ t\in\mathbb{R}_+\end{aligned} \tag{56}$$

由于(56) 式具有引理 2 中不等式(11) 的形式，且相关函数满足引理 2 中的相应条件，我们利用引理 2 就可以得到不等式(56) 中 $m$ 的估计

$$\begin{aligned}m(t)\leqslant&\left(\left(\exp\left(-\left(\ln\left(u(t_0)+\int_{t_0}^{t}A(s)\mathrm{d}s\right)-\int_{t_0}^{t}B(s)\mathrm{d}s\right)\right)-\right.\right.\\&\left.\left.\int_{t_0}^{t}C(s)\mathrm{d}s\right)^2-\int_{t_0}^{t}2D(s)\mathrm{d}s\right)^{-\frac{1}{2}}=M(t),\ t\in[t_0,\infty)\end{aligned} \tag{57}$$

其中 $M(t)$ 由定理 1 中的(36) 式定义. 把(51) 式和(57) 式代入(49) 式可得

$$\dot{r}(t) \leqslant q(t)(1+f(t)+g(t))+[p(t)+f(t)+g(t)+ f(t)p(t)+g(t)p(t)]M(t),\ t \in [t_0,\infty) \tag{58}$$

由(42),(46) 和(58) 式得到

$$r(t) \leqslant u(t_0)+\int_{t_0}^{t}\{q(s)(1+f(s)+g(s))+[p(s)(1+f(s)+g(s))+ f(s)+g(s)]M(s)\}\mathrm{d}s = R(t),\ t \in [t_0,\infty) \tag{59}$$

其中 $R(t)$ 由定理 1 中(35) 式定义. 把(59) 式代入(44) 式可得

$$\dot{z}(t) \leqslant q(t)(1+f(t))+(p(t)+f(t))z(t)+f(t)p(t)R(t),\ t \in [t_0,\infty) \tag{60}$$

即

$$\dot{z}(t)-(p(t)+f(t))z(t) \leqslant q(t)(1+f(t))+f(t)p(t)R(t),\ t \in [t_0,) \tag{61}$$

(61) 式两边同乘$\exp\left(-\int_{t_0}^{t}(p(s)+f(s))\mathrm{d}s\right)$得

$$\frac{\mathrm{d}\left\{z(t)\exp\left(-\int_{t_0}^{t}(p(s)+f(s))\mathrm{d}s\right)\right\}}{\mathrm{d}t} \leqslant (q(t)(1+f(t))+f(t)p(t)R(t))\times \exp\left(-\int_{t_0}^{t}(p(s)+f(s))\mathrm{d}s\right),\ t \in [t_0,\infty) \tag{62}$$

对(62) 式两边积分, 利用(42) 式得到

$$z(t) \leqslant u(t_0)\exp\left(\int_{t_0}^{t}(p(s)+f(s))\mathrm{d}s\right)+\int_{t_0}^{t}[q(s)(1+f(s))+f(s)p(s)R(s)] \exp\left(\int_{s}^{t}(p(\tau)+f(\tau))\mathrm{d}\tau\right)\mathrm{d}s = Z(t),\ t \in [t_0,\infty)$$

$$\dot{u}(t) \leqslant q(t)+p(t)Z(t),\ t \in [t_0,\infty) \tag{63}$$

其中 $Z(t)$ 由定理(1) 中(34) 式定义. 由(63) 式得到定理(1) 所要求的 $u(t)$ 估计式(33).

## 2 应用

本文结果可以用来研究相应类型的微分-积分方程解的性质. 考虑微分-积分方程

$$\dot{x}(t) = q(t)+p(t)\left(x(t)+\int_{t_0}^{t}F(s,x(s),\dot{x}(s))\mathrm{d}s\right),\ x(0)=c \tag{64}$$

**推论 1** 假设方程(64) 中 $|c|$ 是正常数, $q(t)$, $p(t)$ 和定理 1 中 $q(t)$, $p(t)$ 的定义相同. $F \in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ 满足下列条件

$$|F(t,x,y)| \leqslant f(t)\left\{(|x|+|y|)+p(t)\int_{t_0}^{t}|G(s,x,y)|\mathrm{d}s\right\}$$

$$|G(t,x,y)| \leqslant g(t)\left\{(|x|+|y|)+g(t)p(t)\int_{t_0}^{t}|H(s,x,y)|\mathrm{d}s\right\}$$

$$|H(t,x,y)| \leqslant h(t)|y|(|x|^2+|y|^{\theta}) \tag{65}$$

其中: $f(t)$, $g(t)$, $h(t)$ 和 $\theta$ 如定理 1 中的定义. 假设 $|c|$, $q(t)$, $p(t)$, $f(t)$, $g(t)$, $h(t)$ 和 $\theta$ 满足

$$\left(\exp\left(-\left(\ln\left(|c|+\int_{t_0}^{t}A(s)\mathrm{d}s\right)+\int_{t_0}^{t}B(s)\mathrm{d}s\right)\right)-\int_{t_0}^{t}C(s)\mathrm{d}s\right)^2-\int_{t_0}^{t}2D(s)\mathrm{d}s>0,\ t \in [t_0,\infty)$$

如果 $x(t)$ 是方程(64) 的解, 则对于任意 $K>0$, 方程(64) 解的模的估计式

$$|x(t)| \leqslant |c|+\int_{t_0}^{t}(q(s)+p(s)\widetilde{Z}(s))\mathrm{d}s,\ t \in [t_0,\infty) \tag{66}$$

其中

$$\widetilde{Z}(t) := |c|\exp\left(\int_{t_0}^{t}(p(s)+f(s))\mathrm{d}s\right)+\int_{t_0}^{t}[q(s)(1+f(s))+f(s)p(s)\widetilde{R}(s)]$$

$$\exp\left(\int_s^t (p(\tau)+f(\tau))\mathrm{d}\tau\right)\mathrm{d}s$$

$$\widetilde{R}(t):=|c|+\int_{t_0}^t \{q(s)(1+f(s)+g(s))+[p(s)(1+f(s)+g(s))+ f(s)+g(s)]\widetilde{M}(s)\}\mathrm{d}s$$

$$\widetilde{M}(t):=\left(\left(\exp\left(-\left(\ln\left(|c|+\int_{t_0}^t A(s)\mathrm{d}s\right)+\int_{t_0}^t B(s)\mathrm{d}s\right)\right)-\int_{t_0}^t C(s)\mathrm{d}s\right)^2-\int_{t_0}^t 2D(s)\mathrm{d}s\right)^{-\frac{1}{2}}$$

$A(t)$，$B(t)$，$C(t)$，$D(t)$ 由定理 1 中的(37)，(38)，(39) 和(40) 式定义.

**证**　利用条件(65)，由方程(64) 推出

$$|\dot{x}(t)|\leqslant q(t)+p(t)\Big\{|x(t)|+\int_{t_0}^t f(s)(|x(s)|+|\dot{x}(s)|)\mathrm{d}s+ \int_{t_0}^t f(s)p(s)\left(\int_{t_0}^s g(\tau)(|x(\tau)|+|\dot{x}(\tau)|)\mathrm{d}\tau\right)\mathrm{d}s+\int_{t_0}^t f(s)p(s)\times \left[\int_{t_0}^s g(\tau)p(\tau)\left(\int_{t_0}^\tau h(\sigma)|\dot{x}(\sigma)|(x^2(\sigma)+|\dot{x}^\theta(\sigma)|)\mathrm{d}\sigma\right)\mathrm{d}\tau\right]\mathrm{d}s\Big\},\ t\in[t_0,\infty) \tag{67}$$

由于式(67) 具有不等式(9) 的形式，且满足定理 1 中的相应条件，利用定理 1 就可以得到所求的方程解的模的估计式(66).

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# Estimation of Unknown Function of a Class of Triple Integral Inequalities with Unknown Derivative Function

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**Abstract**: Gronwall type integral inequality is the important tool in the study of existence, uniqueness, boundedness and other qualitative properties of solutions of differential equations, integral equation and integro-differential equations. In this paper, a class of nonlinear triple integral inequality is studied, which includes an unknown function and its derivative function in integrand function, and a nonconstant factor outside integral sign. The upper bounds of the unknown function in the integro-differential inequality is estimated explicitly using the techniques of change of variable, the method of amplification, and inverse function technique, which generalized some known results. The derived results can be applied in the study of the explicit upper bounds of solutions of a class of integro-differential equations.

**Key words**: nonlinear integral inequality; triple integral with unknown derivative function; integro-differential equation; explicit estimation

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