DOI:10.13718/j.cnki.xsxb.2020.03.004

关于带自由表面的 Navier-Stokes 方程的一点注记①

田清霞, 朱朝生

西南大学 数学与统计学院, 重庆 400715

摘要: 利用文中的一些定义和复合函数的求导方法以及 Newton-Leibniz 公式, 对 η 函数(一个关键性函数)的一阶和二阶偏导进行了详细推导.

关 键 词: Fourier 变换; 极坐标变换; Newton-Leibniz 公式

中图分类号: O175.29 **文献标志码: A** **文章编号:** 1000-5471(2020)03-0021-06

近年来, Navier-Stokes 方程的相关研究已经引起了广泛的关注[1-2]. 文献[3]考虑了不可压缩粘性流体在高 Reynolds 数下的运动方程(不可压缩 Navier-Stokes 方程):

$$\partial_t u + u \cdot \nabla u + \nabla p = \varepsilon \Delta u \;, \; \nabla \cdot u = 0, \; x \in \Omega_t, \; t > 0$$

其中流体域 $\Omega_t = \{x \in \mathbb{R}^3, \; x_3 < h(t, x_1, x_2)\}$. 文中通过一个微分型同胚映射 $\Phi(t, \cdot)$ 将运动域上的问题简化到固定域 S 上,

$$\Phi(t, \cdot): S = \mathbb{R}^2 \times (-\infty, 0) \longrightarrow \Omega_t,$$
$$x = (y, z) \mapsto (y, \varphi(t, y, z))$$

其中

$$\varphi(t, y, z) = Az + \eta(t, y, z)$$

这里产生的 η 函数是 h 的延拓, 即 $\hat{\eta}(\xi, z) = \chi(z\xi) + \hat{h}(\xi)$, 其中 $\hat{\cdot}$ 表示关于变量 y 的 Fourier 变换, χ 是一个光滑的紧支集函数, 使得在 $B(0, 1)$ 上 $\chi = 1$. 另外常数 $A > 0$, 这使得 $\partial_z \varphi > 0$. 文献[4-9]都研究了这个 η 函数. 文献[3]中多次对 φ 进行求导, 因此不可避免地必须对 η 函数进行求导, 而 η 函数的求导过程较为繁琐, 并且文献[3]并没有给出具体的计算过程, 但是必须有 η 函数的求导结果才有后续的证明结果. 因此本文专门讨论了 η 函数的一阶偏导和二阶偏导.

1 方法示例

设 $F(t) = \int_{x^2+y^2 \leqslant t^2} f(x, y) \mathrm{d}x \mathrm{d}y$, 首先计算 $F'(t), F''(t)$. 为此作变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $0 \leqslant r \leqslant |t|$, $0 \leqslant \theta \leqslant 2\pi$, 则有

$$F(t) = \int_0^{|t|} \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r \mathrm{d}\theta \mathrm{d}r \tag{1}$$

一般地, 我们定义 $G(r) = \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r \mathrm{d}\theta$.

---

① 收稿日期: 2019-01-03
基金项目: 国家自然科学基金项目(11571283).
作者简介: 田清霞(1994-), 女, 硕士研究生, 主要从事偏微分方程现代理论研究.
通信作者: 朱朝生, 副教授.

1) 当 $t>0$ 时，$r\leqslant t$，则 $F(t)=\int_0^t G(r)\mathrm{d}r$，所以

$$F'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t G(r)\mathrm{d}r\right)=G(t)=t\int_0^{2\pi} f(t\cos\theta,\ t\sin\theta)\mathrm{d}\theta \tag{2}$$

$$F''(t)=\int_0^{2\pi} f(t\cos\theta,\ t\sin\theta)\ \mathrm{d}\theta+t\int_0^{2\pi}(\partial_1 f\cos\theta+\partial_2 f\sin\theta)\mathrm{d}\theta \tag{3}$$

2) 类似地，当 $t<0$ 时，$r\leqslant -t$，则 $F(t)=\int_0^{-t} G(r)\mathrm{d}r$，所以

$$F'(t)=-G(t)=-t\int_0^{2\pi} f(t\cos\theta,\ t\sin\theta)\mathrm{d}\theta \tag{4}$$

$$F''(t)=-\int_0^{2\pi} f(t\cos\theta,\ t\sin\theta)\ \mathrm{d}\theta-t\int_0^{2\pi}(\partial_1 f\cos\theta+\partial_2 f\sin\theta)\mathrm{d}\theta \tag{5}$$

3) 因为当 $t=0$ 时，$F(0)=0$，所以当 $t\to 0_+$ 时，

$$\begin{aligned}F'_+(0)&=\lim_{\Delta t\to 0_+}\frac{F(0+\Delta t)-F(0)}{\Delta t}=\lim_{\Delta t\to 0_+}\frac{F(\Delta t)}{\Delta t}=\\&\lim_{\Delta t\to 0_+}\frac{\int_{x^2+y^2\leqslant(\Delta t)^2} f(x,\ y)\mathrm{d}x\mathrm{d}y}{\Delta t}\overset{\text{洛必达}}{=\!=}\lim_{\Delta t\to 0_+}\left(\int_0^{\Delta t} G(r)\mathrm{d}r\right)'_{\Delta t}=\\&\lim_{\Delta t\to 0_+} G(\Delta t)=\lim_{\Delta t\to 0_+}\Delta t\int_0^{2\pi} f(\Delta t\cos\theta,\Delta t\sin\theta)\mathrm{d}\theta=0\end{aligned} \tag{6}$$

类似地，当 $t\to 0_-$ 时，易得 $F'_-(0)=0$.

2 函数 $\boldsymbol{\eta}(t,\ \boldsymbol{y},\ z)$ 的二阶偏导

下面引入 Fourier 变换和 Fourier 逆变换的定义：

$$\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}\mathrm{d}x \qquad f(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\mathrm{d}\xi$$

由文献[4-6]知

$$\begin{aligned}\eta(t,\ \boldsymbol{y},\ z)&=\int_{\mathbb{R}^2}\hat{\eta}(t,\ \boldsymbol{\xi},\ z)\mathrm{e}^{2\mathrm{i}\pi\boldsymbol{y}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}=\int_{\mathbb{R}^2}\chi(z\boldsymbol{\xi})\hat{h}(t,\ \boldsymbol{\xi})\mathrm{e}^{2\mathrm{i}\pi\boldsymbol{y}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}=\\&\int_{|z\boldsymbol{\xi}|\leqslant 1}\hat{h}(t,\ \boldsymbol{\xi})\mathrm{e}^{2\mathrm{i}\pi\boldsymbol{y}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}+\int_{1\leqslant|z\boldsymbol{\xi}|\leqslant R}\chi(z\boldsymbol{\xi})\hat{h}(t,\ \boldsymbol{\xi})\mathrm{e}^{2\mathrm{i}\pi\boldsymbol{y}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}=:\\&\Phi(t,\ \boldsymbol{y},\ z)+\Psi(t,\ \boldsymbol{y},\ z)\end{aligned}$$

其中：$\boldsymbol{y}=(y_1,\ y_2)$，$\boldsymbol{\xi}=(\xi_1,\ \xi_2)$．这里 $\hat{\cdot}$ 表示关于变量 $\boldsymbol{y}$ 的 Fourier 变换，R 表示圆域的半径，χ 是一个光滑的紧支集函数，使得在 $B(0,1)$ 上 $\chi=1$.

下面讨论 $\Psi(t,\ \boldsymbol{y},\ z)$．由于 $1\leqslant|z\boldsymbol{\xi}|\leqslant R$，所以 $1\leqslant z^2\xi_1{}^2+z^2\xi_2{}^2\leqslant R^2$，则 $\frac{1}{z^2}\leqslant\xi_1^2+\xi_2^2\leqslant\frac{R^2}{z^2}$．因此作变换 $\begin{cases}\xi_1=r\cos\theta\\ \xi_2=r\sin\theta\end{cases}$，$\frac{1}{|z|}\leqslant r\leqslant\frac{R}{|z|}$，$0\leqslant\theta\leqslant 2\pi$．为了方便记：

$$\bar{\chi}:=\chi(\cos\theta,\ \sin\theta),\ \bar{\chi}:=\chi(R\cos\theta,\ R\sin\theta),\ \chi:=\chi(zr\cos\theta,\ zr\sin\theta)$$

$$\bar{H}:=\hat{h}\left(t,\ \frac{1}{z}\cos\theta,\ \frac{1}{z}\sin\theta\right),\ \tilde{H}:=\hat{h}\left(t,\ \frac{R}{z}\cos\theta,\ \frac{R}{z}\sin\theta\right),\ H:=\hat{h}(t,\ r\cos\theta,\ r\sin\theta)$$

$$\bar{E}:=\mathrm{e}^{\frac{2\mathrm{i}\pi}{z}(y_1\cos\theta+y_2\sin\theta)},\ \tilde{E}:=\mathrm{e}^{\frac{2\mathrm{i}\pi R}{z}(y_1\cos\theta+y_2\sin\theta)},\ E:=\mathrm{e}^{2\mathrm{i}\pi r(y_1\cos\theta+y_2\sin\theta)}$$

则有

$$\Psi(t,\ \boldsymbol{y},\ z)=\int_{\frac{1}{|z|}}^{\frac{R}{|z|}}\int_0^{2\pi}\chi\hat{h}Er\mathrm{d}\theta\mathrm{d}r=\int_{\frac{1}{|z|}}^{\frac{R}{|z|}} f(t,\ \boldsymbol{y},\ z,\ r)\mathrm{d}r \tag{7}$$

其中 $f(t,\ \boldsymbol{y},\ z,\ r)=\int_0^{2\pi}\chi(zr\cos\theta,\ zr\sin\theta)\hat{h}(t,\ r\cos\theta,\ r\sin\theta)\mathrm{e}^{2\mathrm{i}\pi r(y_1\cos\theta+y_2\sin\theta)}r\mathrm{d}\theta$.

命题 1 当 $z>0$ 时，

$$\Psi''_{zy_1}(t,\boldsymbol{y},z)=-\frac{2\mathrm{i}\pi R^3}{z^4}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\widetilde{E}\cos\theta\mathrm{d}\theta+\frac{2\mathrm{i}\pi}{z^4}\int_0^{2\pi}\bar{\chi}\overline{H}\overline{E}\cos\theta\mathrm{d}\theta+ 2\mathrm{i}\pi\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta+\partial_2\chi\sin\theta)HEr^3\cos\theta\mathrm{d}\theta\mathrm{d}r \tag{8}$$

$$\Psi''_{zy_2}(t,\boldsymbol{y},z)=-\frac{2\mathrm{i}\pi R^3}{z^4}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\widetilde{E}\sin\theta\mathrm{d}\theta+\frac{2\mathrm{i}\pi}{z^4}\int_0^{2\pi}\bar{\chi}\overline{H}\overline{E}\sin\theta\mathrm{d}\theta+ 2\mathrm{i}\pi\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta+\partial_2\chi\sin\theta)HEr^3\sin\theta\mathrm{d}\theta\mathrm{d}r \tag{9}$$

$$\Psi''_{zt}(t,\boldsymbol{y},z)=-\frac{R^2}{z^3}\int_0^{2\pi}\tilde{\chi}\partial_t\widetilde{H}\widetilde{E}\mathrm{d}\theta+\frac{1}{z^3}\int_0^{2\pi}\bar{\chi}\partial_t\overline{H}\overline{E}\mathrm{d}\theta+ \int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta+\partial_2\chi\sin\theta)\partial_t HEr^2\mathrm{d}\theta\mathrm{d}r \tag{10}$$

$$\begin{aligned}\Psi''_{zz}(t,\boldsymbol{y},z)=&\frac{3R^2}{z^4}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\widetilde{E}\mathrm{d}\theta+\frac{R^3}{z^5}\int_0^{2\pi}(\partial_2\widetilde{H}\cos\theta+\partial_3\widetilde{H}\sin\theta)\tilde{\chi}\widetilde{E}\mathrm{d}\theta+\\&\frac{2\mathrm{i}\pi R^3}{z^5}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\widetilde{E}(y_1\cos\theta+y_2\sin\theta)\mathrm{d}\theta-\frac{3}{z^4}\int_0^{2\pi}\bar{\chi}\overline{H}\overline{E}\mathrm{d}\theta-\\&\frac{1}{z^5}\int_0^{2\pi}(\partial_2\overline{H}\cos\theta+\partial_3\overline{H}\sin\theta)\bar{\chi}\overline{E}\mathrm{d}\theta-\frac{2\mathrm{i}\pi}{z^5}\int_0^{2\pi}\bar{\chi}\overline{H}\overline{E}(y_1\cos\theta+y_2\sin\theta)\mathrm{d}\theta-\\&\frac{R^3}{z^4}\int_0^{2\pi}[\partial_1\chi(R\cos\theta,R\sin\theta)\cos\theta+\partial_2\chi(R\cos\theta,R\sin\theta)\sin\theta]\widetilde{H}\widetilde{E}\mathrm{d}\theta\mathrm{d}r+\\&\frac{1}{z^4}\int_0^{2\pi}[\partial_1\chi(\cos\theta,\sin\theta)\cos\theta+\partial_2\chi(\cos\theta,\sin\theta)\sin\theta]\overline{H}\overline{E}\mathrm{d}\theta\mathrm{d}r+\\&\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}[\partial_{11}\chi\cos^2\theta+\partial_{12}\chi\sin\theta\cos\theta+\partial_{21}\chi\sin\theta\cos\theta+\partial_{22}\chi\sin^2\theta]HEr^3\mathrm{d}\theta\mathrm{d}r\end{aligned} \tag{11}$$

证　当 $z>0$ 时，$\frac{1}{z}\leqslant r\leqslant\frac{R}{z}$，则 $\Psi(t,\boldsymbol{y},z)=\int_{\frac{1}{z}}^{\frac{R}{z}}f(t,\boldsymbol{y},z,r)\mathrm{d}r$. 所以，由文献[10]的 Newton-Leibniz 公式可得：

$$\begin{aligned}\Psi'_z(t,\boldsymbol{y},z)=&f\left(t,\boldsymbol{y},z,\frac{R}{z}\right)\left(\frac{R}{z}\right)'-f\left(t,\boldsymbol{y},z,\frac{1}{z}\right)\left(\frac{1}{z}\right)'+\int_{\frac{1}{z}}^{\frac{R}{z}}f'_z(t,\boldsymbol{y},z,r)\mathrm{d}r=\\&-\frac{R}{z^2}\int_0^{2\pi}\chi\left(z\frac{R}{z}\cos\theta,z\frac{R}{z}\sin\theta\right)\hat{h}\left(t,\frac{R}{z}\cos\theta,\frac{R}{z}\sin\theta\right)\mathrm{e}^{\frac{2\mathrm{i}\pi R}{z}(y_1\cos\theta+y_2\sin\theta)}\frac{R}{z}\mathrm{d}\theta+\\&\frac{1}{z^2}\int_0^{2\pi}\chi\left(z\frac{1}{z}\cos\theta,z\frac{1}{z}\sin\theta\right)\hat{h}\left(t,\frac{1}{z}\cos\theta,\frac{1}{z}\sin\theta\right)\mathrm{e}^{\frac{2\mathrm{i}\pi}{z}(y_1\cos\theta+y_2\sin\theta)}\frac{1}{z}\mathrm{d}\theta+\\&\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}[\partial_1\chi(zr\cos\theta,zr\sin\theta)r\cos\theta+\partial_2\chi(zr\cos\theta,zr\sin\theta)r\sin\theta]\times\\&[\hat{h}(t,r\cos\theta,r\sin\theta)\mathrm{e}^{2\mathrm{i}\pi r(y_1\cos\theta+y_2\sin\theta)}r]\mathrm{d}\theta\mathrm{d}r=\\&-\frac{R^2}{z^3}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\widetilde{E}\mathrm{d}\theta+\frac{1}{z^3}\int_0^{2\pi}\bar{\chi}\overline{H}\overline{E}\mathrm{d}\theta+\\&\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta+\partial_2\chi\sin\theta)HEr^2\mathrm{d}\theta\mathrm{d}r\end{aligned} \tag{12}$$

进一步求导得：

$$\Psi''_{zy_1}(t,\boldsymbol{y},z)=-\frac{R^2}{z^3}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\frac{2\mathrm{i}\pi R}{z}\cos\theta\widetilde{E}\mathrm{d}\theta+\frac{1}{z^3}\int_0^{2\pi}\bar{\chi}\overline{H}\frac{2\mathrm{i}\pi}{z}\cos\theta\overline{E}\mathrm{d}\theta+ \int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta+\partial_2\chi\sin\theta)HE(2\mathrm{i}\pi)r^3\cos\theta\mathrm{d}\theta\mathrm{d}r$$

由此容易得到(8)式. 同理可得方程(9)和(10). 下面计算 $\Psi''_{zz}(t,\boldsymbol{y},z)$，

$$\Psi''_{zz}(t, \boldsymbol{y}, z) = \frac{3R^2}{z^4}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\widetilde{E}\,\mathrm{d}\theta - \frac{R^2}{z^3}\int_0^{2\pi}\tilde{\chi}(\partial_2\widetilde{H}\cos\theta + \partial_3\widetilde{H}\sin\theta)\left(-\frac{R}{z^2}\right)\widetilde{E}\mathrm{d}\theta -$$
$$\frac{R^2}{z^3}\int_0^{2\pi}\tilde{\chi}\widetilde{H}\left(-\frac{2\mathrm{i}\pi R}{z^2}\right)(y_1\cos\theta + y_2\sin\theta)\widetilde{E}\mathrm{d}\theta -$$
$$\frac{3}{z^4}\int_0^{2\pi}\bar{\chi}\overline{H}\overline{E}\,\mathrm{d}\theta + \frac{1}{z^3}\int_0^{2\pi}\bar{\chi}(\partial_2\overline{H}\cos\theta + \partial_3\overline{H}\sin\theta)\left(-\frac{1}{z^2}\right)\overline{E}d\theta +$$
$$\frac{1}{z^3}\int_0^{2\pi}\bar{\chi}\overline{H}\left(-\frac{2\mathrm{i}\pi}{z^2}\right)(y_1\cos\theta + y_2\sin\theta)\overline{E}\mathrm{d}\theta +$$
$$\left[-\frac{R}{z^2}\int_0^{2\pi}[\partial_1\chi(R\cos\theta, R\sin\theta)\cos\theta + \partial_2\chi(R\cos\theta, R\sin\theta)\sin\theta]\widetilde{H}\widetilde{E}\left(\frac{R}{z}\right)^2\mathrm{d}\theta\mathrm{d}r +\right.$$
$$\frac{1}{z^2}\int_0^{2\pi}[\partial_1\chi(\cos\theta, \sin\theta)\cos\theta + \partial_2\chi(\cos\theta, \sin\theta)\sin\theta]\overline{H}\overline{E}\left(\frac{1}{z}\right)^2\mathrm{d}\theta\mathrm{d}r +$$
$$\left.\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}[\partial_{11}\chi r\cos^2\theta + \partial_{12}\chi r\sin\theta\cos\theta + \partial_{21}\chi r\sin\theta\cos\theta + \partial_{22}\chi r\sin^2\theta]HEr^2\mathrm{d}\theta\mathrm{d}r\right]$$

由此容易得到(11)式.

下面讨论当 $z<0$ 时的情形，为了方便记：

$$\bar{\chi}_-:=\chi(-\cos\theta, -\sin\theta) \qquad \tilde{\chi}_-:=\chi(-R\cos\theta, -R\sin\theta)$$
$$\overline{H}_-:=\hat{h}\left(t, -\frac{1}{z}\cos\theta, -\frac{1}{z}\sin\theta\right) \qquad \widetilde{H}_-:=\hat{h}\left(t, -\frac{R}{z}\cos\theta, -\frac{R}{z}\sin\theta\right)$$
$$\overline{E}_-:=\mathrm{e}^{-\frac{2\mathrm{i}\pi}{z}(y_1\cos\theta + y_2\sin\theta)} \qquad \widetilde{E}_-:=\mathrm{e}^{-\frac{2\mathrm{i}\pi R}{z}(y_1\cos\theta + y_2\sin\theta)}$$

命题 2 当 $z<0$ 时，

$$\Psi''_{zy_1}(t, \boldsymbol{y}, z) = -\frac{2\mathrm{i}\pi R^3}{z^5}\int_0^{2\pi}\tilde{\chi}_-\widetilde{H}_-\widetilde{E}_-\cos\theta\mathrm{d}\theta + \frac{2\mathrm{i}\pi}{z^5}\int_0^{2\pi}\bar{\chi}_-\overline{H}_-\overline{E}_-\cos\theta\mathrm{d}\theta +$$
$$2\mathrm{i}\pi\int_{-\frac{1}{z}}^{-\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta + \partial_2\chi\sin\theta)HEr^3\cos\theta\mathrm{d}\theta\mathrm{d}r \tag{13}$$

$$\Psi''_{zy_2}(t, \boldsymbol{y}, z) = -\frac{2\mathrm{i}\pi R^3}{z^5}\int_0^{2\pi}\tilde{\chi}_-\widetilde{H}_-\widetilde{E}_-\sin\theta\mathrm{d}\theta + \frac{2\mathrm{i}\pi}{z^5}\int_0^{2\pi}\bar{\chi}_-\overline{H}_-\overline{E}_-\sin\theta\mathrm{d}\theta +$$
$$2\mathrm{i}\pi\int_{-\frac{1}{z}}^{-\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta + \partial_2\chi\sin\theta)HEr^3\sin\theta\mathrm{d}\theta\mathrm{d}r \tag{14}$$

$$\Psi''_{zt}(t, \boldsymbol{y}, z) = -\frac{R^2}{z^3}\int_0^{2\pi}\chi_-\partial_t\widetilde{H}_-\widetilde{E}_-\mathrm{d}\theta + \frac{1}{z^3}\int_0^{2\pi}\bar{\chi}_-\partial_t\overline{H}_-\overline{E}_-\mathrm{d}\theta +$$
$$\int_{-\frac{1}{z}}^{-\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta + \partial_2\chi\sin\theta)\partial_t HEr^2\mathrm{d}\theta\mathrm{d}r \tag{15}$$

$$\Psi''_{zz}(t, \boldsymbol{y}, z) = \frac{3R^2}{z^4}\int_0^{2\pi}\tilde{\chi}_-\widetilde{H}_-\widetilde{E}_-\mathrm{d}\theta - \frac{R^3}{z^5}\int_0^{2\pi}(\partial_2\widetilde{H}_-\cos\theta + \partial_3\widetilde{H}_-\sin\theta)\tilde{\chi}_-\widetilde{E}_-\mathrm{d}\theta -$$
$$\frac{2\mathrm{i}\pi R^3}{z^5}\int_0^{2\pi}\tilde{\chi}_-\widetilde{H}_-\widetilde{E}_-(y_1\cos\theta + y_2\sin\theta)\mathrm{d}\theta - \frac{3}{z^4}\int_0^{2\pi}\bar{\chi}_-\overline{H}_-\overline{E}_-\mathrm{d}\theta +$$
$$\frac{1}{z^5}\int_0^{2\pi}(\partial_2\overline{H}_-\cos\theta + \partial_3\overline{H}_-\sin\theta)\bar{\chi}_-\overline{E}_-\mathrm{d}\theta + \frac{2\mathrm{i}\pi}{z^5}\int_0^{2\pi}\bar{\chi}_-\overline{H}_-\overline{E}_-(y_1\cos\theta + y_2\sin\theta)\mathrm{d}\theta +$$
$$\frac{R^3}{z^4}\int_0^{2\pi}[\partial_1\chi(-R\cos\theta, -R\sin\theta)\cos\theta + \partial_2\chi(-R\cos\theta, -R\sin\theta)\sin\theta]\widetilde{H}_-\widetilde{E}_-\mathrm{d}\theta\mathrm{d}r -$$
$$\frac{1}{z^4}\int_0^{2\pi}[\partial_1\chi(-\cos\theta, -\sin\theta)\cos\theta + \partial_2\chi(-\cos\theta, -\sin\theta)\sin\theta]\overline{H}_-\overline{E}_-\mathrm{d}\theta\mathrm{d}r +$$
$$\int_{\frac{1}{z}}^{\frac{R}{z}}\int_0^{2\pi}[\partial_{11}\chi\cos^2\theta + \partial_{12}\chi\sin\theta\cos\theta + \partial_{21}\chi\sin\theta\cos\theta + \partial_{22}\chi\sin^2\theta]HEr^3\mathrm{d}\theta\mathrm{d}r \tag{16}$$

证 当 $z<0$ 时，$-\frac{1}{z}\leqslant r\leqslant -\frac{R}{z}$，则 $\boldsymbol{\Psi}(t, \boldsymbol{y}, z) = \int_{-\frac{1}{z}}^{-\frac{R}{z}}f(t, \boldsymbol{y}, z, r)\mathrm{d}r$. 所以易得

$$\Psi'_z(t,\boldsymbol{y},z)=-\frac{R^2}{z^3}\int_0^{2\pi}\tilde{\chi}_-\ \widetilde{H}_-\ \widetilde{E}_-\ \mathrm{d}\theta+\frac{1}{z^3}\int_0^{2\pi}\bar{\chi}_-\ \overline{H}_-\ \overline{E}_-\ \mathrm{d}\theta+$$

$$\int_{-\frac{1}{z}}^{-\frac{R}{z}}\int_0^{2\pi}(\partial_1\chi\cos\theta+\partial_2\chi\sin\theta)HEr^2\mathrm{d}\theta\mathrm{d}r \tag{17}$$

进一步，采用命题 1 的方法进行计算，很容易得到方程(13)-(16).

其次，讨论 $\Phi(t,\boldsymbol{y},z)$. 由于 $|z\boldsymbol{\xi}|\leqslant 1$，所以 $z^2\xi_1{}^2+z^2\xi_2{}^2\leqslant 1$，则 $\xi_1^2+\xi_2^2\leqslant\frac{1}{z^2}=\frac{1}{|z|^2}$. 因此作变换 $\begin{cases}\xi_1=r\cos\theta\\ \xi_2=r\sin\theta\end{cases}$，$0\leqslant r\leqslant\frac{1}{|z|}$，$0\leqslant\theta\leqslant 2\pi$，则有

$$\Phi(t,\boldsymbol{y},z)=\int_0^{\frac{1}{|z|}}\int_0^{2\pi}\hat{h}(t,r\cos\theta,r\sin\theta)\mathrm{e}^{2\mathrm{i}\pi r(y_1\cos\theta+y_2\sin\theta)}r\mathrm{d}\theta\mathrm{d}r=\int_0^{\frac{1}{|z|}}G(t,\boldsymbol{y},r)\mathrm{d}r \tag{18}$$

其中 $G(t,\boldsymbol{y},r)=\int_0^{2\pi}\hat{h}(t,r\cos\theta,r\sin\theta)\mathrm{e}^{2\mathrm{i}\pi r(y_1\cos\theta+y_2\sin\theta)}r\mathrm{d}\theta$. 当 $z\to 0$ 时，$\Phi(t,\boldsymbol{y},z)=\int_{\mathbb{R}^2}\hat{h}(t,\boldsymbol{\xi})\mathrm{e}^{2\mathrm{i}\pi\boldsymbol{y}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}$，容易计算各阶偏导数. 下面分别讨论 $z>0$ 和 $z<0$ 的情形. 为了方便，记 $\hat{h}(t,z\cos\theta,z\sin\theta):=\hat{h}$，$\mathrm{e}^{2\mathrm{i}\pi z(y_1\cos\theta+y_2\sin\theta)}:=\tilde{\mathrm{e}}$.

命题 3　1）当 $z<0$ 时，

$$\Phi''_{zy_1}(t,\boldsymbol{y},z)=-2\mathrm{i}\pi\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\cos\theta\mathrm{d}\theta \tag{19}$$

$$\Phi''_{zy_2}(t,\boldsymbol{y},z)=-2\mathrm{i}\pi\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\sin\theta\mathrm{d}\theta \tag{20}$$

$$\Phi''_{zt}(t,\boldsymbol{y},z)=-\frac{1}{z}\int_0^{2\pi}\partial_t\hat{h}\tilde{\mathrm{e}}\mathrm{d}\theta \tag{21}$$

$$\Phi''_{zz}(t,\boldsymbol{y},z)=\frac{1}{z^2}\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\mathrm{d}\theta-\frac{1}{z}\int_0^{2\pi}(\partial_2\hat{h}\cos\theta+\partial_3\hat{h}\sin\theta)\tilde{\mathrm{e}}\mathrm{d}\theta-$$

$$\frac{2\mathrm{i}\pi}{z}\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}(y_1\cos\theta+y_2\sin\theta)\mathrm{d}\theta \tag{22}$$

2）当 $z<0$ 时，

$$\Phi''_{zy_1}(t,\boldsymbol{y},z)=2\mathrm{i}\pi\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\cos\theta\mathrm{d}\theta \tag{23}$$

$$\Phi''_{zy_2}(t,\boldsymbol{y},z)=2\mathrm{i}\pi\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\sin\theta\mathrm{d}\theta \tag{24}$$

$$\Phi''_{zt}(t,\boldsymbol{y},z)=\frac{1}{z}\int_0^{2\pi}\partial_t\hat{h}\tilde{\mathrm{e}}\mathrm{d}\theta \tag{25}$$

$$\Phi''_{zz}(t,\boldsymbol{y},z)=-\frac{1}{z^2}\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\mathrm{d}\theta+\frac{1}{z}\int_0^{2\pi}(\partial_2\hat{h}\cos\theta+\partial_3\hat{h}\sin\theta)\tilde{\mathrm{e}}\mathrm{d}\theta+$$

$$\frac{2\mathrm{i}\pi}{z}\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}(y_1\cos\theta+y_2\sin\theta)\mathrm{d}\theta \tag{26}$$

证　当 $z>0$ 时，$\Phi(t,\boldsymbol{y},z)=\int_0^{\frac{1}{z}}G(t,\boldsymbol{y},r)\mathrm{d}r$，则 $\Phi'_z(t,\boldsymbol{y},z)=-\frac{1}{z^2}G(t,\boldsymbol{y},z)$. 又因为

$$G(t,\boldsymbol{y},z)=\int_0^{2\pi}\hat{h}(t,z\cos\theta,z\sin\theta)\mathrm{e}^{2\mathrm{i}\pi z(y_1\cos\theta+y_2\sin\theta)}z\mathrm{d}\theta$$

所以 $\Phi'_z(t,\boldsymbol{y},z)=-\frac{1}{z}\int_0^{2\pi}\hat{h}\tilde{\mathrm{e}}\mathrm{d}\theta$. 进一步求导可得方程(19)-(22).

当 $z<0$ 时，$\Phi(t,\boldsymbol{y},z)=\int_0^{-\frac{1}{z}}G(t,\boldsymbol{y},r)\mathrm{d}r$，则 $\Phi'_z(t,\boldsymbol{y},z)=\frac{1}{z^2}G(t,\boldsymbol{y},z)$. 同理可得方程(23)-(26).

参考文献：

[1]　薛菊峰，尚月强. 非定常不可压 Navier-Stokes 方程基于欧拉格式的两水平变分多尺度方法 [J]. 西南大学学报(自然科

学版), 2018, 40(9): 84-90.
[2] 周海军, 高真圣. Navier-Stokes-Poisson 方程的两个注记 [J]. 贵州师范大学学报(自然科学版), 2016, 34(4): 54-57.
[3] MASMOUDI N, ROUSSET F. Uniform Regularity and Vanishing Viscosity Limit for the Free Surface Navier-Stokes Equations [J]. Archive for Rational Mechanics and Analysis, 2017, 223(1): 301-417.
[4] TAREK M E, DONGHYUN L. Uniform Regularity for Free-Boundary Navier-Stokes Equations with Surface Tension [J]. arXiv: Analysis of PDEs, 2014.
[5] LEE D. Uniform Estimate of Viscous Free-Boundary Magnetohydrodynamics with Zero Vacuum Magnetic Field [J]. SIAM Journal on Mathematical Analysis, 2017, 49(4): 2710-2789.
[6] 吴复洲. 旋量层与带自由表面的 Navier-Stokes 方程组的粘性消失极限 [D]. 北京: 清华大学, 2016.
[7] MEI Y, WANG Y, XIN Z P. Uniform Regularity for the Free Surface Compressible Navier-Stokes Equations with or Without Surface Tension [J]. Mathematical Models and Methods in Applied Sciences, 2018, 28(2): 259-336.
[8] GUO Y, TICE I. Local Well-Posedness of the Viscous Surface Wave Problem Without Surface Tension [J]. Analysis & PDE, 2013, 6(2): 287-369.
[9] BEALE J T. The Initial Value Problem for the Navier-Stokes Equations with a Free Surface [J]. Communications on Pure and Applied Mathematics, 1981, 34(3): 359-392.
[10] 华东师范大学数学系编. 数学分析(下册) [M]. 4 版. 北京: 高等教育出版社, 2010: 185-188.

A Note on Navier-Stokes Equation with Free Surface

TIAN Qing-xia, ZHU Chao-sheng
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Abstract: In this paper, the first and second partial derivatives of a key function η have been derived in detail by means of some definitions and derivation methods of composite functions and Newton-Leibniz formula.

Key words: Fourier transform; polar coordinate transformation; Newton-Leibniz formula

责任编辑 张 构