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空间-时间分数阶(2+1)-维 Maccari 方程组的新精确解①

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摘要： 通过扩展的试探方程法去求解空间-时间分数阶(2+1)-维 Maccari 方程组

$$\begin{cases} \mathrm{i}D_t^{\alpha}q + D_{xx}^{2\beta}q + qr = 0 \\ D_t^{\alpha}r + D_{\rho}^{\gamma}r + D_{\tau}^{\beta}(|q|^2) = 0 \end{cases}$$

的精确解，得到了5组新的精确解．这些解分为3类，即有理数解、双曲函数解、指数函数解，极大地丰富了解系，并且这些解在光纤学、量子力学、海洋学和光学等科学中也具有多种应用．

关 键 词： 空间-时间分数阶(2+1)-维 Maccari 方程组；扩展的试探方程法；精确解

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分数阶微积分又称非整数阶微积分，是整数阶微积分的推广．我们可以利用分数阶微积分对具有记忆和遗传性质的材料和过程很好地进行建模，同时也可将其广泛应用于反常扩散、波传播及湍流等不同领域的问题中．17世纪的德国哲学家和数学家 Leibnitz 发明了 Leibnitz 符号，在 Leibnitz 符号下，y 关于 x 的导数可记为$\dfrac{\mathrm{d}x}{\mathrm{d}y}$，相应的 n 阶导数可记为$\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$．随着 n 为分数的问题的提出，数学家 Lacroix 在1819年首次提出了任意阶导数．当前，分数阶导数得到了更大的发展，其中包括 Riemann-Liouville 定义和 Caputo 定义[1-2]．但是，这些定义并不能满足两个函数乘积、商的导数和链规则的已知公式．直到2014年，文献[3]提出了分数阶导数的新定义，称为适形分数导数，记为

$$(D_{\alpha}f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \qquad t > 0,\ \alpha \in (0,1]$$

其性质也被给出．

发展方程是数学理论中的一个重要分支，其中分数阶发展方程在工程、物理和经济法的各个领域中得到了广泛的应用，非线性分数阶微分方程解的存在性问题也被广泛证实[4-7]．近年来，分数阶发展方程的求解也获得了显著的成果，许多整数阶发展方程的求解方法(比如拉普拉斯变换法、上下解法、交替分带并行差分法、分离变量法、同伦分析变换法等[8-11])已经被推广到了分数阶发展方程．1996年，数学家 Maccaei 采用傅立叶展开和时空重构的渐近精确约简方法，从 Kadomtsev-Petviashvili 方程中导出了本文所研究的空间-时间分数阶(2+1)-维 Maccari 方程组

$$\begin{cases} \mathrm{i}D_t^{\alpha}q + D_{xx}^{2\beta}q + qr = 0 \\ D_t^{\alpha}r + D_{\rho}^{\gamma}r + D_{\tau}^{\beta}(|q|^2) = 0 \end{cases}$$

该方程组描述了水波动力学中的无扰动波，在海洋和光学科学领域有着广泛应用．文献[12]利用分数复变

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换将分数阶特殊偏微分方程转化为相应的偏微分方程，采用 $e^{-\varphi(\eta)}$ -展开法得到了双曲函数解、三角函数解、有理函数解及指数函数解. 文献[13]通过 KP 层次约简，得到了亮 N -孤子解及暗 N -孤子解. 文献[14]利用最近提出的扩展试探方程法研究了空间-时间分数阶(2+1)-维 Maccari 方程组，得到了新的孤立波解. 本文将在使用扩展试探方程法的基础上对解的形式作出两种合理改进，得到空间-时间分数阶(2+1)-维 Maccari 方程组的新精确解.

1 扩展试探方程法

对于含独立变量 τ,ρ,t 的非线性方程

$$L(u,\ D_t^{\alpha}u,\ D_{\rho}^{\eta}u,\ D_{\tau}^{\beta}u,\ \cdots,\ D_t^{\alpha}D_t^{\alpha}u,\ D_t^{\alpha}D_{\rho}^{\eta}u,\ D_{\rho}^{\eta}D_{\rho}^{\eta}u,\ D_{\tau}^{\beta}D_{\rho}^{\eta}u,\ \cdots)=0 \qquad \alpha,\beta,\eta\in(0,\ 1] \tag{1}$$

其中 u 是未知函数，L 是 u 及 u 的关于 τ,ρ,t 各阶偏导数的多项式. 使用扩展试探方程法求解方程(1)的主要步骤如下:

步骤 1 通过行波变换将独立变量 τ,ρ,t 转化为行波变量 ξ.

$$u(\tau,\rho,t)=U(\xi) \qquad \xi=\frac{\omega_1\tau^{\beta}}{\beta}+\frac{\omega_2\rho^{\eta}}{\eta}+\frac{\omega_3 t^{\alpha}}{\alpha} \tag{2}$$

其中 $\omega_1,\omega_2,\omega_3$ 是非零变量. 利用(2)式及适形分数导数的性质，方程(1)可转化为只含行波变量 ξ 的常微分方程

$$N(U,\ U',\ U'',\ U''',\ \cdots)=0 \tag{3}$$

其中 N 是含 U 及 U 的关于 ξ 各阶导数的多项式.

步骤 2 假设方程(3)的解可表示为 $U(\xi)=\sum\limits_{k=0}^{m}a_kY^k(\xi)$ 和 $U(\xi)=\sum\limits_{k=-m}^{m}a_kY^k(\xi)$，其中 $a_k(k=-m,-m+1,\cdots,-1,0,1,\cdots,m)$ 是待定常数，正整数 m 由齐次平衡法确定. $Y(\xi)$ 满足方程

$$Y'=h_0+h_1Y+h_2Y^2+h_3Y^3 \tag{4}$$

其中 h_0,h_1,h_2,h_3 是任意常数.

步骤 3 求解方程(4)，得到 Y 的值.

当 $h_3=0$ 时，方程(4)有如下解:

$\Delta=h_1^2-4h_2h_0>0$ 时，

$$Y_1=-\frac{h_1}{2h_2}-\frac{\sqrt{h_1^2-4h_2h_0}}{2h_2}\coth\frac{\sqrt{h_1^2-4h_2h_0}(\xi+\xi_0)}{2}$$

$$Y_2=-\frac{h_1}{2h_2}-\frac{\sqrt{h_1^2-4h_2h_0}}{2h_2}\operatorname{th}\frac{\sqrt{h_1^2-4h_2h_0}(\xi+\xi_0)}{2}$$

$\Delta=h_1^2-4h_2h_0=0$ 时，$Y_3=-\dfrac{1}{h_2(\xi+\xi_0)}-\dfrac{h_1}{2h_2}$;

$\Delta=h_1^2-4h_2h_0<0$ 时，

$$Y_4=-\frac{h_1}{2h_2}-\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}}{2h_2}\coth\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}(\xi+\xi_0)}{2}$$

$$Y_5=-\frac{h_1}{2h_2}-\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}}{2h_2}\operatorname{th}\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}(\xi+\xi_0)}{2}$$

当 $h_0=h_2=0$ 时，方程(4)有如下解:

$$Y_6=\pm\frac{\sqrt{h_1}\exp h_1(\xi+\xi_0)}{\sqrt{1-h_3\exp 2h_1(\xi+\xi_0)}} \qquad h_1>0$$

$$Y_7=\pm\frac{\sqrt{-h_1}\exp h_1(\xi+\xi_0)}{\sqrt{1+h_3\exp 2h_1(\xi+\xi_0)}} \qquad h_1<0$$

当 $h_0=h_1=h_2=0$ 时，方程(4)有如下解:

$$Y_8=\pm\frac{1}{\sqrt{-2h_3(\xi+\xi_0)}}$$

步骤 4　将步骤 2 中的解 $U(\xi)$ 代入方程(3)，运用方程(4)来合并 Y 的相同幂次项，则方程(3)的左端变成一个关于 Y 的多项式. 令该多项式的 Y 的各阶幂次的系数为 0，导出一组关于 $a_k(k=-m,-m+1,\cdots,-1,0,1,\cdots,m)$，$h_j(j=0,1,2,3)$，$\omega_l(l=1,2,3)$，$\theta_1,\theta_2,\theta_3$ 的代数方程组.

步骤 5　求解步骤 4 中建立的代数方程组，将解得的 $a_k(k=-m,-m+1,\cdots,-1,0,1,\cdots,m)$，$h_j(j=0,1,2,3)$，$\omega_3,\theta_3$ 和方程(4)的通解代入到步骤 2 的解 $U(\xi)$ 中，再通过(2)式，即得到方程(1)的解.

2　空间-时间分数阶(2+1)-维 Maccari 方程组的新精确解

空间-时间分数阶(2+1)-维 Maccari 方程组形式如下：

$$\begin{cases} \mathrm{i}D_t^\alpha q + D_{\tau\tau}^{2\beta} q + qr = 0 \\ D_t^\alpha r + D_\rho^\eta r + D_\tau^\beta(|q|^2) = 0 \end{cases} \tag{5}$$

其中 q 是复函数，r 是实函数，τ,ρ 是群速度分量，t 是时间分量[13].

对方程组(5)引入下列行波变换：

$$q(\tau,\rho,t)=Q(\xi)e^{\mathrm{i}\theta} \qquad r(\tau,\rho,t)=R(\xi) \qquad \mathrm{i}=\sqrt{-1}$$

$$\xi=\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{\omega_3 t^\alpha}{\alpha} \qquad \theta=\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\theta_3 t^\alpha}{\alpha}$$

则方程组(5)变为

$$\begin{cases} \mathrm{i}(\omega_3+2\omega_1\theta_1)Q'-(\theta_3+\theta_1^2)Q+\omega_1^2Q''+QR=0 \\ (\omega_2+\omega_3)R'+\omega_1(Q^2)'=0 \end{cases} \tag{6}$$

对方程组(6)中的第二个方程积分，并令积分常数为 0，则得到

$$R=\left(\frac{-\omega_1}{\omega_2+\omega_3}\right)Q^2 \tag{7}$$

将方程组(6)中第一个方程的实部与虚部分离，得到

$$\omega_1^2Q''-(\theta_1^2+\theta_3)Q+QR=0 \tag{8}$$

$$(\omega_3+2\omega_1\theta_1)Q'=0 \tag{9}$$

将方程(7)代入到方程(8)，得到

$$\omega_1^2Q''-(\theta_1^2+\theta_3)Q-\frac{\omega_1}{\omega_2+\omega_3}Q^3=0 \tag{10}$$

在方程(10)中，利用齐次平衡法，得到 $3m=m+4$，即 $m=2$.

情形 1　$Q(\xi)=\sum_{k=0}^{m}a_kY^k(\xi)$，$Y'=h_0+h_1Y+h_2Y^2+h_3Y^3$.

将 $Q(\xi)$ 代入方程(10)进行计算，得到一组关于 $a_k(k=0,1,2)$，$h_j(j=0,1,2,3)$，$\omega_l(l=1,2,3)$，$\theta_1,\theta_2,\theta_3$ 的代数方程组

$$\begin{cases} a_1h_0h_1\omega_1^2+2a_2h_0^2\omega_1^2-\theta_1^2a_0-\theta_3a_0-\dfrac{a_0^3\omega_1}{\omega_2+\omega_3}=0 \\ a_1h_1^2\omega_1^2+6a_2h_0h_1\omega_1^2+2a_1h_0h_2\omega_1^2-\theta_1^2a_1-\theta_3a_1-\dfrac{3a_0^2a_1\omega_1}{\omega_2+\omega_3}=0 \\ 3a_1h_1h_2\omega_1^2+8a_2h_0h_2\omega_1^2+3a_1h_0h_3\omega_1^2+4a_2h_1^2\omega_1^2-\theta_1^2a_2-\theta_3a_2-\dfrac{3a_0^2a_2\omega_1+3a_0a_1^2\omega_1}{\omega_2+\omega_3}=0 \\ 4a_1h_1h_3\omega_1^2+10a_2h_0h_3\omega_1^2+10a_2h_1h_2\omega_1^2+2a_1h_2^2\omega_1^2-\dfrac{6a_0a_1a_2\omega_1+a_1^3\omega_1}{\omega_2+\omega_3}=0 \\ 5a_1h_2h_3\omega_1^2+12a_2h_1h_3\omega_1^2+6a_2h_2^2\omega_1^2-\dfrac{3a_0a_2^2\omega_1+3a_1^2a_2\omega_1}{\omega_2+\omega_3}=0 \\ 3a_1h_3^2\omega_1^2+14a_2h_2h_3\omega_1^2-\dfrac{3a_1a_2^2\omega_1}{\omega_2+\omega_3}=0 \\ 8a_2h_3^2\omega_1^2-\dfrac{a_2^3\omega_1}{\omega_2+\omega_3}=0 \end{cases} \tag{11}$$

通过解代数方程组(11)，得到相应系数 $a_k(k=0,1,2),h_j(j=0,1,2,3),\omega_3,\theta_3$ 的值，即可得到方程组(5)的解：

第1组：$a_0=\pm h_1\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}$，$a_2=\pm 2h_3\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}$，$h_0=h_2=0$，$a_1=0$，$\omega_3=-2\omega_1\theta_1$，$\theta_3=-2\omega_1^2h_1^2-\theta_1^2$.

$$r_1=-2\omega_1^2h_1^2-\frac{8\omega_1^2h_1^2h_3^2\exp\left[4h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}{\left[1-h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]\right]^2}-\frac{8\omega_1^2h_1^2h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}{1-h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}$$

$$q_1=\pm\left[h_1\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}+\frac{2h_1h_3\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}{1-h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}\right]\cdot\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{(-2\omega_1^2h_1^2-\theta_1^2)t^\alpha}{\alpha}\right)\right]$$

或

$$r_2=-2\omega_1^2h_1^2-\frac{8\omega_1^2h_1^2h_3^2\exp\left[4h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}{\left[1+h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]\right]^2}+\frac{8\omega_1^2h_1^2h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}{1+h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}$$

$$q_2=\pm\left[h_1\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}-\frac{2h_1h_3\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}{1+h_3\exp\left[2h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]}\right]\cdot\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{(-2\omega_1^2h_1^2-\theta_1^2)t^\alpha}{\alpha}\right)\right]$$

第2组：$a_2=\pm 2h_3\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}$，$h_0=h_1=h_2=0$，$a_0=a_1=0$，$\omega_3=-2\omega_1\theta_1$，$\theta_3=-\theta_1^2$.

$$r_3=-\frac{2\omega_1^2}{\left[\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right]^2}$$

$$q_3=\pm\frac{\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0}\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{-\theta_1^2t^\alpha}{\alpha}\right)\right]$$

情形2 $Q(\xi)=\sum_{k=-m}^{m}a_kY^k(\xi)$，$Y'=h_0+h_1Y+h_2Y^2+h_3Y^3$.

将 $Q(\xi)$ 代入方程(10)进行计算，得到 $a_k(k=-2,-1,0,1,2),h_j(j=0,1,2,3),\omega_l(l=1,2,3),\theta_1,\theta_2,\theta_3$ 应满足如下的方程：

$$-\frac{a_{-2}^3\omega_1}{\omega_2+\omega_3}=0\qquad -\frac{3a_{-2}^2a_{-1}\omega_1}{\omega_2+\omega_3}=0\qquad 6a_{-2}h_0^2\omega_1^2-\frac{3a_{-2}^2a_0\omega_1+3a_{-2}a_{-1}^2\omega_1}{\omega_2+\omega_3}=0$$

$$10a_{-2}h_0h_1\omega_1^2+2a_{-1}h_0^2\omega_1^2-\frac{3a_{-2}^2a_1\omega_1+a_{-1}^3\omega_1+6a_{-2}a_{-1}a_0\omega_1}{\omega_2+\omega_3}=0$$

$$8a_{-2}h_0h_2\omega_1^2+3a_{-1}h_0h_1\omega_1^2+4a_{-2}h_1^2\omega_1^2-\theta_1^2a_{-2}-\theta_3a_{-2}-\frac{3a_{-2}^2a_2\omega_1+6a_{-2}a_{-1}a_1\omega_1+3a_{-2}a_0^2\omega_1}{\omega_2+\omega_3}-\frac{3a_{-1}^2a_0\omega_1}{\omega_2+\omega_3}=0$$

$$6a_{-2}h_0h_3\omega_1^2+6a_{-2}h_1h_2\omega_1^2+2a_{-1}h_0h_2\omega_1^2+a_{-1}h_1^2\omega_1^2-\theta_1^2a_{-1}-\theta_3a_{-1}-\frac{6a_{-2}a_{-1}a_2\omega_1+6a_{-2}a_0a_1\omega_1}{\omega_2+\omega_3}-\frac{3a_{-1}^2a_1\omega_1+3a_{-1}a_0^2\omega_1}{\omega_2+\omega_3}=0$$

$$4a_{-2}h_1h_3\omega_1^2+a_{-1}h_0h_3\omega_1^2+2a_{-2}h_2^2\omega_1^2+a_{-1}h_1h_2\omega_1^2+a_1h_0h_1\omega_1^2+2a_2h_0^2\omega_1^2-\theta_1^2a_0-\theta_3a_0-\frac{6a_{-2}a_0a_2\omega_1+3a_{-1}^2a_2\omega_1+3a_{-2}a_1^2\omega_1+6a_{-1}a_0a_1\omega_1+a_0^3\omega_1}{\omega_2+\omega_3}=0$$

$$2a_{-2}h_2h_3\omega_1^2+a_1h_1^2\omega_1^2+6a_2h_0h_1\omega_1^2+2a_1h_0h_2\omega_1^2-\theta_1^2a_1-\theta_3a_1-\frac{6a_{-2}a_1a_2\omega_1+6a_{-1}a_0a_2\omega_1}{\omega_2+\omega_3}-\frac{3a_{-1}a_1^2\omega_1+3a_0^2a_1\omega_1}{\omega_2+\omega_3}=0$$

$$-a_{-1}h_2h_3\omega_1^2+3a_1h_1h_2\omega_1^2+8a_2h_0h_2\omega_1^2+3a_1h_0h_3\omega_1^2+4a_2h_1^2\omega_1^2-\theta_1^2a_2-\theta_3a_2-\frac{3a_{-2}a_2^2\omega_1+6a_{-1}a_1a_2\omega_1+3a_0^2a_2\omega_1+3a_0a_1^2\omega_1}{\omega_2+\omega_3}=0$$

$$-a_{-1}h_3^2\omega_1^2+4a_1h_1h_3\omega_1^2+10a_2h_0h_3\omega_1^2+10a_2h_1h_2\omega_1^2+2a_1h_2^2\omega_1^2-\frac{3a_{-1}a_2^2\omega_1+6a_0a_1a_2\omega_1+a_1^3\omega_1}{\omega_2+\omega_3}=0$$

$$5a_1h_2h_3\omega_1^2+12a_2h_1h_3\omega_1^2+6a_2h_2^2\omega_1^2-\frac{3a_0a_2^2\omega_1+3a_1^2a_2\omega_1}{\omega_2+\omega_3}=0,\ 3a_1h_3^2\omega_1^2+14a_2h_2h_3\omega_1^2-\frac{3a_1a_2^2\omega_1}{\omega_2+\omega_3}=0$$

$$8a_2h_3^2\omega_1^2-\frac{a_2^3\omega_1}{\omega_2+\omega_3}=0$$

通过求解这些方程，得到相应系数 $a_k(k=-2,-1,0,1,2)$,$h_j(j=0,1,2,3)$,ω_3,θ_3 的值，即可得到方程组(5)的解：

第 3 组：$a_{-1}=\pm h_0\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}$，$a_0=\pm\frac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}$，$h_3=0$，$a_{-2}=a_1=a_2=0$，$\omega_3=-2\omega_1\theta_1$，$\theta_3=\frac{1}{2}(4h_0h_2\omega_1^2-h_1^2\omega_1^2-2\theta_1^2)$.

$$r_4=-\frac{8h_0^2h_2^2\omega_1^2}{\left[h_1+\sqrt{h_1^2-4h_2h_0}\coth\frac{\sqrt{h_1^2-4h_2h_0}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]^2}+\frac{4h_0h_1h_2\omega_1^2}{h_1+\sqrt{h_1^2-4h_2h_0}\coth\frac{\sqrt{h_1^2-4h_2h_0}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}-\frac{h_1^2\omega_1^2}{2}$$

$$q_4=\pm\left[\frac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}-\frac{2h_0h_2\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{h_1+\sqrt{h_1^2-4h_0h_2}\coth\frac{\sqrt{h_1^2-4h_0h_2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}\right]\cdot\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\frac{1}{2}h_1^2\omega_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_5=-\frac{8h_0^2h_2^2\omega_1^2}{\left[h_1+\sqrt{h_1^2-4h_2h_0}\,\mathrm{th}\frac{\sqrt{h_1^2-4h_2h_0}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]^2}+\frac{4h_0h_1h_2\omega_1^2}{h_1+\sqrt{h_1^2-4h_2h_0}\,\mathrm{th}\frac{\sqrt{h_1^2-4h_2h_0}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}-\frac{h_1^2\omega_1^2}{2}$$

$$q_5=\pm\left[\frac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}-\frac{2h_0h_2\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{h_1+\sqrt{h_1^2-4h_0h_2}\,\mathrm{th}\,\frac{\sqrt{h_1^2-4h_0h_2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\frac{1}{2}h_1^2\omega_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_6=-\frac{8\omega_1^2h_0{}^2h_2{}^2\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)^2}{\left[2+h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\right]^2}+\frac{4\omega_1^2h_0h_1h_2\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2+h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}-\frac{\omega_1^2h_1^2}{2}$$

$$q_6=\pm\left[\frac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}-\frac{2h_0h_2\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{2+h_1\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\frac{1}{2}h_1^2\omega_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_7=-\frac{8h_0^2h_2^2\omega_1^2}{\left[h_1+\mathrm{i}\sqrt{4h_2h_0-h_1^2}\coth\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]^2}+$$
$$\frac{4h_0h_1h_2\omega_1^2}{h_1+\mathrm{i}\sqrt{4h_2h_0-h_1^2}\coth\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}-\frac{h_1^2\omega_1^2}{2}$$

$$q_7=\pm\left[\frac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}-\frac{2h_0h_2\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{h_1+\mathrm{i}\sqrt{4h_0h_2-h_1^2}\coth\frac{\mathrm{i}\sqrt{4h_0h_2-h_1^2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\frac{1}{2}h_1^2\omega_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_8=-\frac{8h_0^2h_2^2\omega_1^2}{\left[h_1+\mathrm{i}\sqrt{4h_2h_0-h_1^2}\,\mathrm{th}\,\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]^2}+$$
$$\frac{4h_0h_1h_2\omega_1^2}{h_1+\mathrm{i}\sqrt{4h_2h_0-h_1^2}\,\mathrm{th}\,\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}-\frac{h_1^2\omega_1^2}{2}$$

$$q_8=\pm\left[\frac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}-\frac{2h_0h_2\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{h_1+\mathrm{i}\sqrt{4h_0h_2-h_1^2}\,\mathrm{th}\,\frac{\mathrm{i}\sqrt{4h_0h_2-h_1^2}\left(\frac{\omega_1\tau^\beta}{\beta}+\frac{\omega_2\rho^\eta}{\eta}+\frac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\frac{1}{2}h_1^2\omega_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

第 4 组：$a_0=\pm\dfrac{h_1\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)}}{\sqrt{2}}$，$a_1=\pm h_2\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}$，$h_3=0$，$a_{-2}=a_{-1}=a_2=0$，$\omega_3=-2\omega_1\theta_1$，$\theta_3=\dfrac{1}{2}(4h_0h_2\omega_1^2-h_1^2\omega_1^2-2\theta_1^2)$.

$$r_9=-\frac{\omega_1^2(h_1^2-4h_2h_0)}{2}\coth^2\frac{\sqrt{h_1^2-4h_2h_0}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}$$

$$q_9=\left[\pm\frac{\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)(h_1^2-4h_2h_0)}}{\sqrt{2}}\coth\frac{\sqrt{h_1^2-4h_2h_0}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\dfrac{1}{2}h_1^2w_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_{10}=-\frac{\omega_1^2(h_1^2-4h_2h_0)}{2}\mathrm{th}^2\frac{\sqrt{h_1^2-4h_2h_0}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}$$

$$q_{10}=\left[\pm\frac{\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)(h_1^2-4h_2h_0)}}{\sqrt{2}}\mathrm{th}\frac{\sqrt{h_1^2-4h_2h_0}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\dfrac{1}{2}h_1^2w_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_{11}=-\frac{2\omega_1^2}{\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)^2}$$

$$q_{11}=\pm\frac{\sqrt{2\omega_1(\omega_2-2\omega_1\theta_1)}}{\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0}\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\dfrac{1}{2}h_1^2\omega_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_{12}=\frac{\omega_1^2(4h_2h_0-h_1^2)}{2}\coth^2\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}$$

$$q_{12}=\left[\pm\frac{\mathrm{i}\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)(4h_2h_0-h_1^2)}}{\sqrt{2}}\coth\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\dfrac{1}{2}h_1^2w_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

或

$$r_{13}=\pm\frac{\omega_1^2(4h_2h_0-h_1^2)}{2}\mathrm{th}^2\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}$$

$$q_{13}=\left[\pm\frac{\mathrm{i}\sqrt{\omega_1(\omega_2-2\omega_1\theta_1)(4h_2h_0-h_1^2)}}{\sqrt{2}}\mathrm{th}\frac{\mathrm{i}\sqrt{4h_2h_0-h_1^2}\left(\dfrac{\omega_1\tau^\beta}{\beta}+\dfrac{\omega_2\rho^\eta}{\eta}+\dfrac{-2\omega_1\theta_1t^\alpha}{\alpha}+\xi_0\right)}{2}\right]\cdot$$
$$\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}+\frac{\left(2h_0h_2\omega_1^2-\dfrac{1}{2}h_1^2w_1^2-\theta_1^2\right)t^\alpha}{\alpha}\right)\right]$$

第5组：$a_0=\pm\dfrac{\sqrt{-\theta_1^2(\omega_2+\omega_3)}}{\sqrt{\omega_1}}$，$h_3=0$，$a_{-2}=a_{-1}=a_1=a_2=0$，$\theta_3=0$.

$$r_{14}=\theta_1^2$$

$$q_{14}=\pm\frac{\sqrt{-\theta_1^2(\omega_2+\omega_3)}}{\sqrt{\omega_1}}\exp\left[\mathrm{i}\left(\frac{\theta_1\tau^\beta}{\beta}+\frac{\theta_2\rho^\eta}{\eta}\right)\right]$$

3 讨论

空间-时间(2+1)-维 Maccari方程组历来受到广泛关注. 多线性变量分离法、广义 F-展开法、扩展扇形方程法、Kadomtsev-Petviashvili(KP)分级还原法等均被用于空间-时间整数阶(2+1)-维 Maccari方程组，获得了周期解、孤子解、双曲函数解、三角函数解等多种形式[13,15-17]. 而随着对空间-时间分数阶(2+1)-维 Maccari方程组的研究的推进，$e^{-\varphi(\eta)}$展开法[12]、扩展试探方程法[14]等解法也被使用. 由于空间-时间分数阶(2+1)-维 Maccari方程组与空间-时间整数阶(2+1)-维 Maccari方程组在方程表达式及行波变换形式上的不同，其得到的解集也不同. 本文在沿用扩展试探方程法的基础上，对解的形式作出了两种合理改进，并得到试探方程 $Y'=h_0+h_1Y+h_2Y^2+h_3Y^3$ 不同的解，使得空间-时间分数阶(2+1)-维 Maccari方程组的解更加多样.

4 结论

本文通过使用扩展试探方程法，经改进后应用于空间-时间分数阶(2+1)-维 Maccari方程组，获得了该方程组5组新的精确解. 这些通解分为3类，即有理数解、双曲函数解和指数函数解，在光纤学、量子力学、海洋学和光学等科学中具有多种应用. 本文拓展了扩展试探方程法的应用，丰富了空间-时间分数阶(2+1)-维Maccari方程组的解系.

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New Exact Solutions of Space-Time Fractional Order (2+1)-Dimensional Maccari Equations

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Abstract: In this paper, the extended trial equation method is used to solve the space-time fractional order (2+1)-dimensional Maccari equations

$$\begin{cases} \mathrm{i}D_t^{\alpha}q + D_{\tau\tau}^{2\beta}q + qr = 0 \\ D_t^{\alpha}r + D_{\rho}^{\eta}r + D_{\tau}^{\beta}(|q|^2) = 0 \end{cases}$$

Five sets of new exact solutions of the space-time fractional order (2+1)-dimensional Maccari equations are obtained. These solutions are divided into three categories: rational, hyperbolic and exponential solutions, which greatly enriched the understanding of the system. And these solutions are also used in optical fiber, quantum mechanics, ocean and optical science.

Key words: space-time fractional order (2+1)-dimensional Maccari equations; extended trial equation method; exact solution

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