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带非线性阻尼项的 Navier-Stokes 方程的时间解析性①

李戈萍, 朱朝生

西南大学 数学与统计学院,重庆 400715

摘要:利用代数化方法处理 Stokes-Ossen 核函数,从而得到了带非线性阻尼项的 Navier-Stokes 方程有界温和解的时间解析性.

关 键 词:时间解析性;Navier-Stokes 方程;温和解;非线性阻尼

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近年来,Navier-Stokes 方程解析性的相关研究已经引起了广泛的关注[1-4].空间解析性是一个局部性质,即在解的某一个领域内讨论其可导性,通常情况下它都是成立的.然而时间解析性就很难说明了.本文代数化地处理 Stokes-Ossen 核函数,就可以得到 $\mathbb{R}^d(d=2,3)$ 内带非线性阻尼项的 Navier-Stokes 方程

$$\begin{cases}\partial_t \boldsymbol{u}-\Delta \boldsymbol{u}+\boldsymbol{u}\cdot\nabla \boldsymbol{u}+|\boldsymbol{u}|^{\beta-1}\boldsymbol{u}+\nabla p=0 & (\boldsymbol{x},t)\in\mathbb{R}^d\times[0,1]\\ \operatorname{div}\boldsymbol{u}=0 & (\boldsymbol{x},t)\in\mathbb{R}^d\times[0,1]\end{cases}\tag{1}$$

其有界温和解的时间解析性.方程(1)中的阻尼项产生于水流运动的阻力,它可以描述一些物理现象,如多孔介质的流动、阻力、摩擦效应以及某些耗散机制[5-7].这里未知函数 $\boldsymbol{u}(\boldsymbol{x},t)$,$p(\boldsymbol{x},t)$ 分别表示不可压缩流体的速度和压强,β 为满足 $\frac{7}{2}\leqslant\beta\leqslant 5$ 的常数.

本文主要结果如下:

定理 1 若问题(1)的温和解 $\boldsymbol{u}$ 满足

$$|\boldsymbol{u}|\leqslant C_1 \qquad (\boldsymbol{x},t)\in\mathbb{R}^d\times[0,1]\tag{2}$$

则对任意整数 $n\geqslant 1$ 有

$$\sup_{t\in(0,1]}\|t^n\partial_t^n\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\leqslant(N+1)^n n^n$$

其中 $N\geqslant 1$ 为充分大的常数.于是,$\boldsymbol{u}(\boldsymbol{x},t)$ 对任意 $t\in(0,1]$ 是时间解析的.

本文中 C_1, C_2, C_3 代表特定的常数,c 代表不定常数.

1 时间解析性

引理 1[9] 对任意 $n\geqslant 1$,下述不等式成立:

$$\sum_{j=1}^{n-1}\binom{n}{j}j^{j-\frac{2}{3}}(n-j)^{n-j-\frac{2}{3}}\leqslant cn^{n-\frac{2}{3}}$$

其中 $c>0$ 为与 n 无关的常数.

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作者简介:李戈萍,硕士研究生,主要从事偏微分方程现代理论研究.

通信作者:朱朝生,副教授.

引理 2[9]　设 f, g 为 $\mathbb{R}$ 上的两个光滑函数，对任意 $n \geqslant 1$，下述关系式成立：

$$\partial_t^n(t^n f(t)g(t)) = t\partial_t^n(t^{n-1}f(t)g(t)) + n\partial_t^{n-1}(t^{n-1}f(t)g(t)) \tag{3}$$

$$\partial_t^n(t^n f(t)g(t)) = \sum_{j=0}^{n}\binom{n}{j}\partial_t^j(t^j f(t))\partial_t^{n-j}(t^{n-j}g(t)) - n\sum_{j=0}^{n-1}\binom{n-1}{j}\partial_t^j(t^j f(t))\partial_t^{n-1-j}(t^{n-1-j}g(t)) \tag{4}$$

令 Stokes-Ossen 核为 $E(\boldsymbol{x}, t) = \widetilde{P}\Gamma(\boldsymbol{x}, t)$，其中 $\widetilde{P}$ 为 $\mathbb{R}^d$ 内的 Leray-Hopf 投影，$\Gamma = (4\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{4t}}$ 为热核. E 为具有半群性质的齐次热方程的解，且 $E(\boldsymbol{x}, t) = t^{-\frac{d}{2}}E\left(\frac{\boldsymbol{x}}{\sqrt{t}}, 1\right)$，其中 $E(\cdot, 1)$ 为 $\mathbb{R}^d$ 内与 $\frac{c}{|\boldsymbol{x}|^d}$ $(\boldsymbol{x}\to\infty)$ 衰减速率相同的光滑函数. 此外，$(\partial_t E)(\boldsymbol{x}, 1)$ 与 $\frac{c}{|\boldsymbol{x}|^{d+2}}(\boldsymbol{x}\to\infty)$ 衰减速率相同，$(\nabla E)(\boldsymbol{x}, 1)$ 与 $\frac{c}{|\boldsymbol{x}|^{d+1}}(\boldsymbol{x}\to\infty)$ 衰减速率相同[13]. 于是，对任意 $t > 0$ 和整数 $k \geqslant 1$ 有

$$\| E(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leqslant C_2 \tag{5}$$

$$\| \nabla E(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leqslant C_2 t^{-\frac{1}{2}} \tag{6}$$

$$\| \partial_t^k E(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leqslant C_2^{k+1}k^{k-\frac{2}{3}}t^{-k} \tag{7}$$

$$\| \partial_t^k \nabla E(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leqslant C_2^{k+1}k^{k-\frac{2}{3}}t^{-k-\frac{1}{2}} \tag{8}$$

其中 $C_2 \geqslant 1$ 为常数. 由 Leibniz 公式[15] 可得

$$\| \partial_t^k(t^k E(\cdot, t))\|_{L^1(\mathbb{R}^d)} \leqslant C_3^{k+1}k^{k-\frac{2}{3}} \tag{9}$$

$$\| \partial_t^k(t^k \nabla E(\cdot, t))\|_{L^1(\mathbb{R}^d)} \leqslant C_3^{k+1}k^{k-\frac{2}{3}}t^{-\frac{1}{2}} \tag{10}$$

其中 $C_3 \geqslant 1$ 为常数.

引理 3　在定理 1 条件成立的前提下，对任意整数 $n \geqslant 1$ 有

$$\sup_{t\in(0, 1]}\| \partial_t^n(t^n\boldsymbol{u}(\cdot, t))\|_{L^\infty(\mathbb{R}^d)} \leqslant N^{n-\frac{1}{2}}\ n^{n-\frac{2}{3}} \tag{11}$$

其中 $N \geqslant 1$ 为充分大的常数.

证　因为 $\boldsymbol{u}$ 是问题(1) 的温和解，所以对任意 $t \in (0, 1]$ 有

$$\boldsymbol{u}(\boldsymbol{x}, t) = E(\boldsymbol{x}, t) * \boldsymbol{u}(\boldsymbol{x}, 0) - \int_0^t E(\boldsymbol{x}, t-\tau) * [\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{x}, \tau) + (|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, \tau)]\mathrm{d}\tau$$

其中 $*$ 为空间卷积. 因为

$$E(\boldsymbol{x}, t-\tau) * \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{x}, \tau) = \int_{\mathbb{R}^d} E(\boldsymbol{x}-\boldsymbol{y}, t-\tau)(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{y}, \tau))\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^d}(\nabla_x E(\boldsymbol{x}-\boldsymbol{y}, t-\tau))(\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{y}, \tau)\mathrm{d}\boldsymbol{y} = \nabla E(\boldsymbol{x}, t-\tau) * (\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{x}, \tau)$$

所以

$$\partial_t^n(t^n\boldsymbol{u}(\boldsymbol{x}, t)) = \partial_t^n(t^n E(\boldsymbol{x}, t) * \boldsymbol{u}(\boldsymbol{x}, 0)) - \partial_t^n\left(\int_0^t t^n\nabla E(\boldsymbol{x}, t-\tau) * (\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{x}, \tau)\mathrm{d}\tau\right) - \partial_t^n\left(\int_0^t t^n E(\boldsymbol{x}, t-\tau) * (|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, \tau)\mathrm{d}\tau\right) := \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3 \tag{12}$$

由(2) 和(9) 式，可得

$$|\boldsymbol{I}_1| = |\partial_t^n(t^n E(\boldsymbol{x}, t) * \boldsymbol{u}(\boldsymbol{x}, 0))| \leqslant \sup_{\boldsymbol{x}\in\mathbb{R}^d}[|\boldsymbol{u}(\boldsymbol{x}, 0)|]\| \partial_t^n(t^n E(\cdot, t))\|_{L^1(\mathbb{R}^d)} \leqslant C_1 C_3^{n+1}n^{n-\frac{2}{3}} \leqslant N^{n-\frac{2}{3}}n^{n-\frac{2}{3}} \tag{13}$$

其中 $N \geqslant 1$ 为充分大的常数. 估计 $\boldsymbol{I}_2$ 之前，先将其作一个简单的变形，即

$$\int_0^t t^n\nabla E(\boldsymbol{x}, t-\tau) * (\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{x}, \tau)\mathrm{d}\tau =$$

$$\int_0^t (t-\tau+\tau)^n \nabla E(\boldsymbol{x}, t-\tau) * (\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, \tau)\mathrm{d}\tau =$$

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^t (t-\tau)^k \nabla E(\boldsymbol{x}, t-\tau) * (\tau^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, \tau))\mathrm{d}\tau$$

所以

$$\boldsymbol{I}_2 = -\sum_{k=0}^{n}\binom{n}{k}\partial_t^n\int_0^t (t-\tau)^k \nabla E(\boldsymbol{x}, t-\tau) * (\tau^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, \tau))\mathrm{d}\tau =$$

$$-\sum_{k=0}^{n}\binom{n}{k}\partial_t^{n-k}\int_0^t \partial_t^k((t-\tau)^k \nabla E(\boldsymbol{x}, t-\tau)) * (\tau^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, \tau))\mathrm{d}\tau =$$

$$-\sum_{k=0}^{n}\binom{n}{k}\partial_t^{n-k}\int_0^t \partial_\tau^k(\tau^k \nabla E(\boldsymbol{x}, \tau)) * ((t-\tau)^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau =$$

$$-\sum_{k=0}^{n}\binom{n}{k}\int_0^t \partial_\tau^k(\tau^k \nabla E(\boldsymbol{x}, \tau)) * \partial_t^{n-k}((t-\tau)^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau$$

由(2)和(4)式可得对任意整数 $k=1,2,\cdots,n-1$ 有

$$|\partial_t^k(t^k(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t))| \leqslant N^{k-\frac{1}{3}}k^{k-\frac{2}{3}} \tag{14}$$

以及 $k=n$ 时有

$$|\partial_t^n(t^n(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t))| \leqslant 2C_1|\partial_t^n(t^n\boldsymbol{u}(\boldsymbol{x}, t))| + N^{n-\frac{3}{4}}n^{n-\frac{2}{3}} \tag{15}$$

其中 $N \geqslant 1$ 为充分大的常数. 于是

$$\boldsymbol{I}_2 = -\sum_{k=0}^{n}\binom{n}{k}\int_0^t \partial_\tau^k(\tau^k \nabla E(\boldsymbol{x}, \tau)) * \partial_t^{n-k}((t-\tau)^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau =$$

$$-\sum_{k=0}^{n-1}\binom{n}{k}\int_0^t \partial_\tau^k(\tau^k \nabla E(\boldsymbol{x}, \tau)) * \partial_t^{n-k}((t-\tau)^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau -$$

$$\int_0^t \nabla E(\boldsymbol{x}, \tau) * \partial_t^n((t-\tau)^n(\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau -$$

$$\int_0^t \partial_\tau^n(\tau^n \nabla E(\boldsymbol{x}, \tau)) * (\boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}, t-\tau)\mathrm{d}\tau := $$

$$\boldsymbol{K}_1 + \boldsymbol{K}_2 + \boldsymbol{K}_3$$

由(2),(6),(10),(14),(15)式和引理1, 可得

$$|\boldsymbol{K}_1| \leqslant \sum_{k=0}^{n-1}\binom{n}{k}\int_0^t \|\partial_\tau^k(\tau^k \nabla E(\cdot, \tau))\|_{L^1(\mathbb{R}^d)} \|\partial_t^{n-k}((t-\tau)^{n-k}(\boldsymbol{u} \otimes \boldsymbol{u})(\cdot, t-\tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \leqslant$$

$$\sum_{k=0}^{n-1}\binom{n}{k}N^{n-k-\frac{1}{3}}(n-k)^{n-k-\frac{2}{3}}C_3^{k+1}k^{k-\frac{2}{3}}\int_0^t \tau^{-\frac{1}{2}}\mathrm{d}\tau \leqslant 2cn^{n-\frac{2}{3}}N^{n-k-\frac{1}{3}}C_3^{k+1}t^{\frac{1}{2}} \leqslant N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t^{\frac{1}{2}}$$

$$|\boldsymbol{K}_2| \leqslant \int_0^t \|\nabla E(\cdot, \tau)\|_{L^1(\mathbb{R}^d)} \|\partial_t^n((t-\tau)^n(\boldsymbol{u} \otimes \boldsymbol{u})(\cdot, t-\tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \leqslant$$

$$2C_1C_2\int_0^t \tau^{-\frac{1}{2}}\|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau + N^{n-\frac{3}{4}}n^{n-\frac{2}{3}}C_2\int_0^t \tau^{-\frac{1}{2}}\mathrm{d}\tau \leqslant$$

$$2C_1C_2\int_0^t \tau^{-\frac{1}{2}}\|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau + N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t^{\frac{1}{2}}$$

$$|\boldsymbol{K}_3| \leqslant [\sup_{\boldsymbol{x}\in\mathbb{R}^d}|\boldsymbol{u}(\cdot, t)|]^2\int_0^t \|\partial_\tau^n(\tau^n \nabla E(\cdot, \tau))\|_{L^1(\mathbb{R}^d)}\mathrm{d}\tau \leqslant$$

$$C_1^2C_3^{n+1}n^{n-\frac{2}{3}}\int_0^t \tau^{-\frac{1}{2}}\mathrm{d}\tau \leqslant N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t^{\frac{1}{2}}$$

其中 $N \geqslant 1$ 为充分大的常数. 所以

$$|\boldsymbol{I}_2| \leqslant N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t^{\frac{1}{2}} + 2C_1C_2\int_0^t \tau^{-\frac{1}{2}}\|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \tag{16}$$

与估计 $\boldsymbol{I}_2$ 同理, 先将 $\boldsymbol{I}_3$ 作一个简单的变形, 即

$$\int_0^t t^n E(\boldsymbol{x}, t-\tau) * (|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, \tau)\mathrm{d}\tau =$$
$$\sum_{k=0}^{n}\binom{n}{k}\int_0^t (t-\tau)^k E(\boldsymbol{x}, t-\tau) * (\tau^{n-k}(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, \tau))\mathrm{d}\tau$$

于是

$$\begin{aligned}\boldsymbol{I}_3 =& -\sum_{k=0}^{n}\binom{n}{k}\partial_t^n\int_0^t (t-\tau)^k E(\boldsymbol{x}, t-\tau) * (\tau^{n-k}(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, \tau))\mathrm{d}\tau = \\ & -\sum_{k=0}^{n}\binom{n}{k}\int_0^t \partial_\tau^k(\tau^k E(\boldsymbol{x}, \tau)) * \partial_t^{n-k}((t-\tau)^{n-k}(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau = \\ & -\sum_{k=0}^{n-1}\binom{n}{k}\int_0^t \partial_\tau^k(\tau^k E(\boldsymbol{x}, \tau)) * \partial_t^{n-k}((t-\tau)^{n-k}(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau - \\ & \int_0^t E(\boldsymbol{x}, \tau) * \partial_t^n((t-\tau)^n(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, t-\tau))\mathrm{d}\tau - \\ & \int_0^t \partial_\tau^n(\tau^n E(\boldsymbol{x}, \tau)) * (|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, t-\tau)\mathrm{d}\tau := \\ & \boldsymbol{K}_4 + \boldsymbol{K}_5 + \boldsymbol{K}_6\end{aligned}$$

由(2) 和(4) 式可得对任意整数 $k = 1,2,\cdots,n-1$ 有

$$|\partial_t^k(t^k(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, t))| \leqslant cC_1^\beta k^{k-\frac{2}{3}} \leqslant N^{k-\frac{1}{3}}k^{k-\frac{2}{3}} \tag{17}$$

以及 $k = n$ 时有

$$|\partial_t^n(t^n(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\boldsymbol{x}, t))| \leqslant C_1^{\beta-1}|\partial_t^n(t^n\boldsymbol{u}(\boldsymbol{x}, t))| \tag{18}$$

其中 $N \geqslant 1$ 为充分大的常数．由(2),(5),(9),(17),(18) 式和引理 1 可得

$$\begin{aligned}|\boldsymbol{K}_4| \leqslant & \sum_{k=0}^{n-1}\binom{n}{k}\int_0^t \|\partial_\tau^k(\tau^k E(\cdot, \tau))\|_{L^1(\mathbb{R}^d)}\|\partial_t^{n-k}((t-\tau)^{n-k}(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\cdot, t-\tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \leqslant \\ & \sum_{k=0}^{n-1}\binom{n}{k}N^{n-k-\frac{1}{3}}(n-k)^{n-k-\frac{2}{3}}C_3^{k+1}k^{k-\frac{2}{3}}\int_0^t 1\mathrm{d}\tau \leqslant \\ & cn^{n-\frac{2}{3}}N^{n-k-\frac{1}{3}}C_3^{k+1}t \leqslant \\ & N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t\end{aligned}$$

$$\begin{aligned}|\boldsymbol{K}_5| \leqslant & \int_0^t \|E(\cdot, \tau)\|_{L^1(\mathbb{R}^d)}\|\partial_t^n((t-\tau)^n(|\boldsymbol{u}|^{\beta-1}\boldsymbol{u})(\cdot, t-\tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \leqslant \\ & C_2C_1^{\beta-1}\int_0^t \|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau\end{aligned}$$

$$\begin{aligned}|\boldsymbol{K}_6| \leqslant & [\sup_{\boldsymbol{x}\in\mathbb{R}^d}|\boldsymbol{u}(\cdot, t)|]^\beta\int_0^t \|\partial_\tau^n(\tau^n E(\cdot, \tau))\|_{L^1(\mathbb{R}^d)}\mathrm{d}\tau \leqslant \\ & C_1^\beta C_3^{n+1}n^{n-\frac{2}{3}}\int_0^t 1\mathrm{d}\tau \leqslant \\ & N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t\end{aligned}$$

其中 $N \geqslant 1$ 为充分大的常数，所以

$$|\boldsymbol{I}_3| \leqslant N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t + C_2C_1^{\beta-1}\int_0^t \|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \tag{19}$$

将(13),(16) 和(19) 式代入(12) 式，可得对任意 $t \in (0, 1]$ 有

$$\begin{aligned}|\partial_t^n(t^n\boldsymbol{u}(\boldsymbol{x}, t))| \leqslant & N^{n-\frac{2}{3}}n^{n-\frac{2}{3}} + N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t^{\frac{1}{2}} + N^{n-\frac{2}{3}}n^{n-\frac{2}{3}}t + \\ & \int_0^t (2C_1C_2\tau^{-\frac{1}{2}} + C_2C_1^{\beta-1})\|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau \leqslant \\ & N^{n-\frac{2}{3}}n^{n-\frac{2}{3}} + \int_0^t (2C_1C_2\tau^{-\frac{1}{2}} + C_2C_1^{\beta-1})\|\partial_\tau^n(\tau^n\boldsymbol{u}(\cdot, \tau))\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}\tau\end{aligned} \tag{20}$$

对(20) 式利用 Gronwall 不等式，可得对任意 $t \in (0, 1]$ 有

$$|\partial_t^n(t^n\boldsymbol{u}(\boldsymbol{x},t))| \leqslant N^{n-\frac{2}{3}} n^{n-\frac{2}{3}} \mathrm{e}^{4C_1C_2+C_2C_1^{\beta-1}} \leqslant N^{n-\frac{1}{2}}\ n^{n-\frac{2}{3}}$$

即

$$\sup_{t\in(0,1]} \|\partial_t^n(t^n\boldsymbol{u}(\cdot,t))\|_{L^\infty(\mathbb{R}^d)} \leqslant N^{n-\frac{1}{2}} n^{n-\frac{2}{3}}$$

其中 $N \geqslant 1$ 为充分大的常数. 综上，引理 3 证明完成.

定理 1 的证明 由(3)式可得对任意 $t \in (0,1]$ 和整数 $k = 1,2,\cdots,n$ 有

$$\partial_t^n(t^k\boldsymbol{u}) = t\partial_t^n(t^{k-1}\boldsymbol{u}) + n\partial_t^{n-1}(t^{k-1}\boldsymbol{u})$$

当 $k = n$ 时有

$$\partial_t^n(t^n\boldsymbol{u}) = t\partial_t^n(t^{n-1}\boldsymbol{u}) + n\partial_t^{n-1}(t^{n-1}\boldsymbol{u})$$

再利用(11)式，可得

$$\begin{aligned} |t\partial_t^n(t^{n-1}\boldsymbol{u})| &= |\partial_t^n(t^n\boldsymbol{u}) - n\partial_t^{n-1}(t^{n-1}\boldsymbol{u})| \leqslant \\ & N^{n-\frac{1}{2}} n^{n-\frac{2}{3}} + nN^{n-1-\frac{1}{2}}(n-1)^{n-1-\frac{2}{3}} \leqslant \\ & \left(1+\frac{1}{N}\right)N^n n^n \end{aligned} \tag{21}$$

当 $k = n-1$ 时有

$$\partial_t^n(t^{n-1}\boldsymbol{u}) = t\partial_t^n(t^{n-2}\boldsymbol{u}) + n\partial_t^{n-1}(t^{n-2}\boldsymbol{u})$$

再利用(21)式可得

$$\begin{aligned} |t^2\partial_t^n(t^{n-2}\boldsymbol{u})| &= |t\partial_t^n(t^{n-1}\boldsymbol{u}) - nt\partial_t^{n-1}(t^{n-2}\boldsymbol{u})| \leqslant \\ & \left(1+\frac{1}{N}\right)N^n n^n + nt\left(1+\frac{1}{N}\right)N^{n-1}(n-1)^{n-1} \leqslant \\ & \left(1+\frac{1}{N}\right)^2 N^n n^n \end{aligned}$$

由归纳法可得当 $k = 1$ 时有

$$|t^n\partial_t^n\boldsymbol{u}| \leqslant \left(1+\frac{1}{N}\right)^n N^n n^n = (N+1)^n n^n$$

即

$$\sup_{t\in(0,1]} \|t^n\partial_t^n\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leqslant (N+1)^n n^n$$

其中 $N \geqslant 1$ 为充分大的常数. 综上，定理 1 证明完成.

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Time Analyticity for the Navier-Stokes Equations with Nonlinear Damping

LI Ge-ping，　ZHU Chao-sheng

School of Mathematics and Statistics，Southwest University，Chongqing 400715，China

Abstract：In this paper，the algebraic method has been used to deal with the Stokes-Ossen kernel. Then，we prove the time analyticity for the bounded mild solutions of the Navier-Stokes equations with nonlinear damping.

Key words：time analyticity；Navier-Stokes equations；mild solutions；nonlinear damping

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