DOI:10.13718/j.cnki.xsxb.2021.07.001

三阶中立型分布时滞微分方程的振动性①

刘 俊[1], 刘 曦[2], 朱春艳[3], 俞元洪[4]

1. 曲靖师范学院 应用数学研究所,云南 曲靖 655011;2. 云南大学 信息学院,昆明 650091;
3. 曲靖师范学院 信息工程学院,云南 曲靖 655011;4. 中国科学院 数学与系统科学研究院,北京 100190

摘要: 对一类三阶中立型分布时滞微分方程进行了研究,研究了 $\beta \geqslant \alpha$ 和 $\beta < \alpha$ 的情况,运用广义 Riccati 变换和特殊技术,获得了该方程解振动或收敛于零的新的充分条件.

关 键 词: 振动准则;微分方程;广义 Riccati 变换;分布时滞

中图分类号:O175.1 **文献标志码:A** **文章编号:** 1000-5471(2021)07-0001-08

考虑如下一类三阶中立型分布时滞微分方程:

$$(r(t)(z''(t))^{\alpha})' + \int_c^d q(t,\xi) f(x(\sigma(t,\xi)))\mathrm{d}\xi = 0 \qquad t \geqslant t_0 > 0 \tag{1}$$

目前对三阶微分方程振动性研究还比较少. 文献[1-3]研究了 $\alpha = \beta$ 的情况,其中文献[3]研究了

$$((x''(t))^{\alpha})' + q(t)x^{\alpha}(\sigma(t)) = 0$$

的振动性. 文献[4]在

$$\frac{f(u)}{u} \geqslant \delta > 0$$

的情形下,研究了

$$(r(t)[x(t) + p(t)x(\tau(t))]'')' + \int_a^b q(t,\xi) f[x(g(t,\xi))]\mathrm{d}\sigma(\xi) = 0$$

的振动性($\alpha = \beta = 1$). 文献[5]在

$$\frac{F(t, x, u, v, w)}{u^{\alpha}} \geqslant q(t)$$

情形,研究了如下三阶半线性中立型微分方程的振动性($\alpha = \beta$):

$$(r(t)([x(t) + p(t)x(\tau(t))]'')^{\alpha})' + F(t, x(t), x(g(t)), x'(t), x'(g(t))) = 0$$

文献[6]在

$$\frac{f(x)}{x^{\alpha}} \geqslant 1$$

情形下,研究了如下三阶微分方程的振动性($\alpha = \beta$):

$$(r(t)([x(t) + \int_a^b p(t,\xi)x(\tau(t,\xi))\mathrm{d}\xi]'')^{\alpha})' + \int_c^d q(t,\xi) f[x(\sigma(t,\xi))]\mathrm{d}\xi = 0$$

文献[7]在 $uf(u) \geqslant 0$ 情形下,研究了

$$(r_2(t)[r_1(t)y']')' + p(t)y' + q(t)f(y(g(t))) = 0$$

---

① 收稿日期:2020-02-24
基金项目:国家自然科学基金项目(11361048);云南省教育厅科学研究项目(2019J0613);曲靖市教育科学研究项目(QJQSKT2019XZC05,QJQSKT2019YB27).
作者简介:刘 俊,教授,硕士,主要从事微分方程和数学教学研究.

振动性($\alpha=\beta=1$). 文献[8-9]在$\alpha=\beta$、文献[10-11]在$\alpha=\beta=1$情形下研究了三阶微分方程的振动性. 文献[12]在$\alpha\geqslant\beta>0$、文献[13]在$\alpha\geqslant\beta$与$\beta\geqslant\alpha$情形研究了二阶微分方程

$$(r(t)\mid z'(t)\mid^{\alpha-1}z'(t))'+q(t)\mid x(\sigma(t))\mid^{\beta-1}x(\sigma(t))=0$$

的振动性. 文献[14]在$\alpha=\beta=1$、文献[15]在$\alpha=\beta$情形研究了三阶微分方程的振动性. 文献[16-20]研究了一些方程或方程组解的存在性及渐近性态.

从上述文献可看到, 三阶微分方程基本上是在$\alpha=\beta$的情形下展开研究的[1-11,14-15], 二阶微分方程也只有极少数文章在$\alpha\geqslant\beta>0$, $\beta\geqslant\alpha$和$\alpha>\beta$的情形下展开研究[12-13].

本文拟在$\beta\geqslant\alpha$和$\alpha>\beta$两种情形下探讨一类三阶中立型分布时滞微分方程(1)的振动性, 运用广义Riccati变换和特殊技术, 同时处理了$\beta\geqslant\alpha$和$\alpha>\beta$两种情形, 并将这两种情形统一起来, 最终获得了方程(1)解振动或收敛于零的新的充分条件, 文章所得的新的结论是上述文献的推广和改进.

在方程(1)中

$$z(t)=x(t)-\int_a^b p(t,\xi)x(\tau(t,\xi))\mathrm{d}\xi$$

并假设下列条件成立:

(Ⅰ) $f(x)\in C(R,R)$, $\dfrac{f(x)}{x^{\beta}}\geqslant\delta>0$, $x\neq0$, α和β是正奇数.

(Ⅱ) $r(t)\in C^1([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(t)\mathrm{d}t=\infty$, $p(t,\xi)\in C([t_0,\infty)\times[a,b],R)$, $p(t,\xi)\geqslant0$, $0\leqslant\int_a^b p(t,\xi)\mathrm{d}\xi\leqslant p_0<1$;

(Ⅲ) $\tau(t,\xi)\in C([t_0,\infty)\times[a,b],R)$, $\tau(t,\xi)\leqslant t$, $\liminf\limits_{t\to\infty}\tau(t,\xi)=\infty$; $\sigma(t,\xi)\in C^1([t_0,\infty)\times[c,d],R)$是关于$\xi$的非减函数, $\sigma(t,\xi)\leqslant t$, $\lim\limits_{t\to\infty}\sigma(t,\xi)=\infty$, $\xi\in[c,d]$;

(Ⅳ) $q(t,\xi)\in C([t_0,\infty]\times[c,d],\mathbb{R}_+)$.

引理1 假设$x(t)$是方程(1)的正解, 且条件(Ⅱ)-(Ⅳ)成立, 则存在$t_1\geqslant t_0$, 使得对$t\geqslant t_1$, $z(t)$具有下列4种可能性质:

(ⅰ) $z(t)>0$, $z'(t)>0$, $z''(t)>0$, $(r(t)\mid z''(t)\mid^{\alpha-1}z''(t))'\leqslant0$;

(ⅱ) $z(t)>0$, $z'(t)<0$, $z''(t)>0$, $(r(t)\mid z''(t)\mid^{\alpha-1}z''(t))'\leqslant0$;

(ⅲ) $z(t)<0$, $z'(t)<0$, $z''(t)>0$, $(r(t)\mid z''(t)\mid^{\alpha-1}z''(t))'\leqslant0$;

(ⅳ) $z(t)<0$, $z'(t)<0$, $z''(t)<0$, $(r(t)\mid z''(t)\mid^{\alpha-1}z''(t))'\leqslant0$.

证 若$x(t)$是方程(1)的正解, 那么存在$t_1\geqslant t_0$, 使得对$t\geqslant t_1$, 有

$$x(t)>0,\ x(\tau(t,\xi))>0,\ \xi\in[a,b]$$

$$x(\sigma(t,\mu))>0,\ \mu\in[c,d]$$

由$z(t)$的定义可得$x(t)\geqslant z(t)$, 从方程(1)得到

$$(r(t)(z''(t))^{\alpha})'=-\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))\mathrm{d}\xi\leqslant0 \tag{2}$$

则$r(t)(z''(t))^{\alpha}$是减函数而且最终定号, 因此, 存在$t_2\geqslant t_1$, 使得$t\geqslant t_2$, 有$z''(t)<0$或$z''(t)>0$.

若$z''(t)<0$, $t\geqslant t_2$, 那么由(2)式得到

$$(r(t)(-z''(t))^{\alpha})'\geqslant0$$

于是$r(t)(-z''(t))^{\alpha}$是增函数, 则存在正常数$M>0$, 使得

$$r(t)(-z''(t))^{\alpha}\geqslant M$$

即

$$z''(t)\leqslant-M^{\frac{1}{\alpha}}r^{-\frac{1}{\alpha}}(t)$$

进一步在$[t_2,t]$上积分, 得到

$$z'(t)\leqslant z'(t_2)-M^{\frac{1}{\alpha}}\int_{t_2}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s$$

由(Ⅱ)知$\lim\limits_{t\to\infty} z'(t)=-\infty$，则$z'(t)<0$最终成立，而$z''(t)<0$，因此$z(t)<0$，得到性质(ⅳ).

若$z''(t)>0$，$t\geqslant t_2$，则$z'(t)$定号. 如果$z'(t)>0$，那么$z(t)>0$，得到性质(ⅰ). 如果$z'(t)<0$，那么$z(t)>0$或$z(t)<0$，得到(ⅱ)和(ⅲ)两种性质.

引理 2　假设$x(t)$是方程(1)的正解，并且$z(t)$满足引理 1 的性质(i)，则存在$t_1\geqslant t_0$，$t_2\geqslant t_1$，使得对$t\geqslant t_2$，有

$$\frac{z(t)}{z'(t)}\geqslant\frac{\int_{t_2}^{t}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\tag{3}$$

且$\dfrac{z'(t)}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}$是减函数.

证　若$z(t)$满足引理 1 性质(i)，则

$$z'(t)=z'(t_1)+\int_{t_1}^{t}z''(s)\mathrm{d}s=z'(t_1)+\int_{t_1}^{t}\frac{(r(s)(z''(s))^{\alpha})^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\mathrm{d}s\geqslant r^{\frac{1}{\alpha}}(t)z''(t)\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s$$

于是

$$\left(\frac{z'(t)}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)'=\frac{z''(t)\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s-z'(t)r^{-\frac{1}{\alpha}}(t)}{\left(\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^2}\leqslant$$

$$\frac{z''(t)\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s-r^{-\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)z''(t)\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}{\left(\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^2}=0$$

则$\dfrac{z'(t)}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}$是减函数，故

$$z(t)=z(t_2)+\int_{t_2}^{t}z'(s)\mathrm{d}s=z(t_2)+\int_{t_2}^{t}\frac{z'(s)}{\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s\geqslant$$

$$\frac{z'(t)}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\cdot\int_{t_2}^{t}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s$$

即

$$\frac{z(t)}{z'(t)}\geqslant\frac{\int_{t_2}^{t}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}$$

定理 1　假设存在函数$\rho(t)\in C^1([t_0,\infty),\mathbb{R}_+)$，使得

$$\limsup_{t\to\infty}\int_{T}^{t}\left(\delta\rho(s)K(s)Q(s)-\frac{1}{(\lambda+1)^{\lambda+1}}\frac{r(s)(\rho'(s))^{\lambda+1}}{\rho^{\lambda}(s)}\right)\mathrm{d}s=\infty\tag{4}$$

其中

$$Q(t)=\int_{c}^{d}q(t,\xi)\mathrm{d}\xi,\ \lambda=\min\{\alpha,\beta\}$$

$$K(t)=\min\{K_1(t),K_2(t)\},\ \sigma_1(t)=\sigma(t,c)$$

$$K_1(t)=M_1\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\alpha}$$

$$K_2(t)=\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s} r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\beta}$$

则引理 1 中的性质(ⅰ)不成立.

证 设 $x(t)$ 是方程(1)的正解，$z(t)$ 满足引理 1 中的性质(i)，由方程(1)得到

$$\begin{aligned}(r(t)(z''(t))^{\alpha})' &=-\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))\mathrm{d}\xi\leqslant-\delta\int_c^d q(t,\xi)x^{\beta}(\sigma(t,\xi))\mathrm{d}\xi\leqslant\\ &-\delta\int_c^d q(t,\xi)z^{\beta}(\sigma(t,\xi))\mathrm{d}\xi\leqslant-\delta z^{\beta}(\sigma(t,c))\int_c^d q(t,\xi)\mathrm{d}\xi=\\ &-\delta Q(t)z^{\beta}(\sigma_1(t))\end{aligned}$$

其中

$$Q(t)=\int_c^d q(t,\xi)\mathrm{d}\xi,\ \sigma_1(t)=\sigma(t,c)$$

分两种情况讨论如下：

1) 若 $\beta\geqslant\alpha$，定义函数如下

$$w(t)=\rho(t)\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}$$

则 $w(t)>0$，且

$$\begin{aligned}w'(t)=&\rho'(t)\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}+\rho(t)\left(\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}\right)'=\\ &\frac{\rho'(t)}{\rho(t)}\frac{\rho(t)r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}+\rho(t)\cdot\frac{(r(t)(z''(t))^{\alpha})'(z'(t))^{\alpha}-r(t)(z''(t))^{\alpha}\cdot\alpha(z'(t))^{\alpha-1}z''(t)}{(z'(t))^{2\alpha}}\leqslant\\ &\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\cdot\frac{-\delta Q(t)z^{\beta}(\sigma_1(t))}{(z'(t))^{\alpha}}-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}\frac{(\rho(t))^{\frac{\alpha+1}{\alpha}}(r(t))^{\frac{\alpha+1}{\alpha}}(z''(t))^{\alpha+1}}{(z'(t))^{\alpha+1}}=\\ &\frac{\rho'(t)}{\rho(t)}w(t)-\delta\rho(t)Q(t)\left(\frac{z(\sigma_1(t)}{z'(t)}\right)^{\alpha}(z(\sigma_1(t))^{\beta-\alpha}-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t)\end{aligned}$$

由于 $z'(t)>0$，则 $z(t)$ 是增函数，而 $z(t)>0$，$\beta\geqslant\alpha$，则存在正常数 $M_1>0$，使得

$$(z(\sigma_1(t))^{\beta-\alpha}\geqslant M_1$$

由引理 2 和条件(Ⅲ)有

$$\frac{z'(\sigma_1(t))}{\int_{t_1}^{\sigma_1(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\geqslant\frac{z'(t)}{\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}$$

即

$$\frac{z'(\sigma_1(t))}{z'(t)}\geqslant\frac{\int_{t_1}^{\sigma_1(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}{\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}$$

于是

$$\begin{aligned}\left(\frac{z(\sigma_1(t))}{z'(t)}\right)^{\alpha}&=\left(\frac{z(\sigma_1(t))}{z'(\sigma_1(t))}\cdot\frac{z'(\sigma_1(t))}{z'(t)}\right)^{\alpha}\geqslant\\ &\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s} r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{\sigma_1(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\cdot\frac{\int_{t_1}^{\sigma_1(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}{\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\alpha}=\\ &M_1\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s} r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\alpha}\cdot\frac{1}{M_1}=\frac{K_1(t)}{M_1}\end{aligned}\tag{5}$$

其中

$$K_1(t)=M_1\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\alpha}$$

因此

$$w'(t)\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\delta\rho(t)Q(t)K_1(t)-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t)\tag{6}$$

2）若 $\beta<\alpha$，定义函数

$$w(t)=\rho(t)\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\beta}}$$

则 $w(t)>0$，并有

$$\begin{aligned}w'(t)=&\rho'(t)\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\beta}}+\rho(t)\left(\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\beta}}\right)'=\\&\frac{\rho'(t)}{\rho(t)}\frac{\rho(t)r(t)(z''(t))^{\alpha}}{(z'(t))^{\beta}}+\rho(t)\cdot\frac{(r(t)(z''(t))^{\alpha})'(z'(t))^{\beta}-r(t)(z''(t))^{\alpha}\cdot\beta(z'(t))^{\beta-1}z''(t)}{(z'(t))^{2\beta}}\leqslant\\&\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\cdot\frac{-\delta Q(t)z^{\beta}(\sigma_1(t))}{(z'(t))^{\beta}}-\frac{\beta(z''(t))^{1-\frac{\alpha}{\beta}}}{(\rho(t)r(t))^{\frac{1}{\beta}}}\frac{(\rho(t))^{1+\frac{1}{\beta}}(r(t))^{1+\frac{1}{\beta}}((z''(t))^{\alpha})^{1+\frac{1}{\beta}}}{((z'(t))^{\beta})^{1+\frac{1}{\beta}}}=\\&\frac{\rho'(t)}{\rho(t)}w(t)-\delta\rho(t)Q(t)\left(\frac{z(\sigma_1(t))}{z'(t)}\right)^{\beta}-\frac{\beta}{(\rho(t)r(t))^{\frac{1}{\beta}}}\cdot\frac{1}{(z''(t))^{\frac{\alpha}{\beta}-1}}w^{\frac{\beta+1}{\beta}}(t)\end{aligned}$$

由方程(1) 和引理 1 的性质(ⅰ)，有 $z'''(t)\leqslant 0$ 即 $z''(t)$ 是减函数，$\frac{1}{z''(t)}$ 是增函数，由于

$$z''(t)>0,\ \frac{\alpha}{\beta}-1>0$$

则存在常数 $M_2\geqslant 1$，使得

$$\frac{1}{(z''(t))^{\frac{\alpha}{\beta}-1}}=\left(\frac{1}{z''(t)}\right)^{\frac{\alpha}{\beta}-1}\geqslant M_2$$

类似于(5) 式得到

$$\begin{aligned}\left(\frac{z(\sigma_1(t))}{z'(t)}\right)^{\beta}=\left(\frac{z(\sigma_1(t))}{z'(\sigma_1(t))}\cdot\frac{z'(\sigma_1(t))}{z'(t)}\right)^{\beta}\geqslant&\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\cdot\frac{\int_{t_1}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\beta}=\\&\left(\frac{\int_{t_2}^{\sigma_1(t)}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\beta}=K_2(t)\end{aligned}$$

于是

$$\begin{aligned}w'(t)\leqslant&\frac{\rho'(t)}{\rho(t)}w(t)-\delta\rho(t)Q(t)K_2(t)-\frac{\beta M_2}{(\rho(t)r(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t)\leqslant\\&\frac{\rho'(t)}{\rho(t)}w(t)-\delta\rho(t)Q(t)K_2(t)-\frac{\beta}{(\rho(t)r(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t)\end{aligned}\tag{7}$$

令

$$K(t)=\min\{K_1(t),\ K_2(t)\}$$
$$\lambda=\min\{\alpha,\ \beta\}$$

则可将(6) 式与(7) 式统一写成

$$w'(t)\leqslant-\delta\rho(t)Q(t)K(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\lambda}{(\rho(t)r(t))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(t)\tag{8}$$

参照文献[1]使用不等式:

$$Bu - Au^{\frac{\lambda+1}{\lambda}} \leqslant \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}} \cdot \frac{B^{\lambda+1}}{A^{\lambda}}$$

则由(8)式得到

$$w'(t) \leqslant -\delta\rho(t)K(t)Q(t) + \frac{1}{(\lambda+1)^{\lambda+1}} \frac{r(t)\,(\rho'(t))^{\lambda+1}}{\rho^{\lambda}(t)}$$

进一步在$[T, t]$上积分,有

$$\int_T^t \left(\delta\rho(s)K(s)Q(s) - \frac{1}{(\lambda+1)^{\lambda+1}} \frac{r(s)\,(\rho'(s))^{\lambda+1}}{\rho^{\lambda}(s)}\right)\mathrm{d}s \leqslant w(T)$$

此式与(4)式相矛盾,则引理1中的性质(i)不存在.

定理 2 假设 $x(t)$ 是方程(1)的正解,且引理1中的性质(ⅱ)成立,若

$$\int_{t_0}^{\infty}\int_v^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty}\int_c^d q(s, \xi)\mathrm{d}\xi\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}u\mathrm{d}v = \infty \tag{9}$$

则$\lim\limits_{t\to\infty} x(t) = 0$.

证 由引理1(ⅱ)知,存在常数 $c \geqslant 0$,使得$\lim\limits_{t\to\infty} z(t) = c$,我们的目标是 $c = 0$. 假设 $c > 0$,由 $z(t)$ 的定义和(ⅱ)知 $x(t) \geqslant z(t) > c$,由方程(1)有

$$(r(t)(z''(t))^{\alpha})' = -\int_c^d q(t, \xi) f(x(\sigma(t, \xi)))\mathrm{d}\xi \leqslant -\delta\int_c^d q(t, \xi) x^{\beta}(\sigma(t, \xi))\mathrm{d}\xi \leqslant$$

$$-\delta\int_c^d q(t, \xi) z^{\beta}(\sigma(t, \xi))\mathrm{d}\xi \leqslant -c^{\beta}\delta\int_c^d q(t, \xi)\mathrm{d}\xi$$

进一步在$[t, \infty]$上积分

$$\int_t^{\infty}(r(s)(z''(s))^{\alpha})'\mathrm{d}s \leqslant -c^{\beta}\delta\int_t^{\infty}\int_c^d q(s, \xi)\mathrm{d}\xi\mathrm{d}s$$

即

$$z''(t) \geqslant c^{\frac{\beta}{\alpha}}\delta^{\frac{1}{\alpha}}\left(\frac{1}{r(t)}\int_t^{\infty}\int_c^d q(s, \xi)\mathrm{d}\xi\mathrm{d}s\right)^{\frac{1}{\alpha}} \tag{10}$$

将(10)式在$[t, \infty]$上积分,再在$[t_1, \infty]$上积分,注意到 $z'(t) < 0$,得到

$$z(t_1) \geqslant c^{\frac{\beta}{\alpha}}\delta^{\frac{1}{\alpha}}\int_{t_1}^{\infty}\int_v^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty}\int_c^d q(s, \xi)\mathrm{d}\xi\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}u\mathrm{d}v$$

与(9)式相矛盾,因此,$c = 0$,即$\lim\limits_{t\to\infty} z(t) = 0$.

接下来,要证明 $x(t)$ 有界. 用反证法,假设 $x(t)$ 无界,则存在序列$\{t_n\}$,使得$\lim\limits_{n\to\infty} t_n = \infty$,$\lim\limits_{n\to\infty} x(t_n) = \infty$. 其中 $x(t_n) = \max\{x(s): t_0 \leqslant s \leqslant t_n\}$.

由条件(Ⅲ),$\lim\limits_{t\to\infty}\inf\tau(t, \xi) = \infty$,可得 $\tau(t_n, \xi) > t_0$ 对充分大的 n 成立. 又由条件(Ⅲ)的 $\tau(t, \xi) \leqslant t$,有

$$x(\tau(t_n, \xi)) = \max\{x(s): t_0 \leqslant s \leqslant \tau(t_n, \xi)\} \leqslant \max\{x(s): t_0 \leqslant s \leqslant t_n\} = x(t_n)$$

则由条件(Ⅱ),有

$$z(t_n) = x(t_n) - \int_a^b p(t_n, \xi)x(\tau(t_n, \xi))\mathrm{d}\xi \geqslant x(t_n) - \int_a^b p(t_n, \xi)x(t_n)\mathrm{d}\xi \geqslant x(t_n)(1 - p_0)$$

由假设$\lim\limits_{t\to\infty} x(t_n) = \infty$ 可得$\lim\limits_{t\to\infty} z(t_n) = \infty$,这与$\lim\limits_{t\to\infty} z(t) = 0$ 相矛盾,所以 $x(t)$ 有界.

设$\lim\limits_{t\to\infty}\sup x(t) = a_0$,$0 \leqslant a_0 < \infty$,则存在序列$\{t_k\}$,使得$\lim\limits_{k\to\infty} t_k = \infty$,$\lim\limits_{k\to\infty} x(t_k) = a_0$. 如果 $a_0 > 0$,则取

$$\varepsilon_0 = \frac{a_0(1 - p_0)}{5p_0} > 0$$

可得 $x(t_k) < a_0 + \varepsilon_0$,我们有

$$0 = \lim_{k\to\infty} z(t_k) \geqslant \lim_{k\to\infty}[x(t_k) - p_0 x(t_k)] \geqslant a_0 - p_0(a_0 + \varepsilon_0) = \frac{4a_0(1 - p_0)}{5} > 0$$

此式矛盾，则 $a_0 = 0$，故$\lim\limits_{t\to\infty} x(t) = 0$.

定理 3　假设(4) 式和(9) 式成立，则方程(1) 的解振动或者$\lim\limits_{t\to\infty} x(t) = 0$.

证　由定理 1 知，引理 1 性质(ⅰ)不存在. 若引理 1 性质(ⅱ)满足，由定理 2 得到$\lim\limits_{t\to\infty} x(t) = 0$.

如果引理 1 的性质(ⅲ)或(ⅳ)满足，则$\lim\limits_{t\to\infty} z(t) = c_0 < 0$(可能 $c_0 = -\infty$) 或$\lim\limits_{t\to\infty} z(t) = -\infty$. 类似于定理 2 的推导可知 $x(t)$ 和 $z(t)$ 有界，于是 c_0 是有限数，则性质(ⅳ)不会满足，类似于定理 2 的推导可得到$\lim\limits_{t\to\infty} x(t) = 0$. 因此，方程(1) 的解振动或$\lim\limits_{t\to\infty} x(t) = 0$.

例　考虑下列三阶微分方程

$$\left(x(t) - \frac{1}{2}\int_0^{\frac{\pi}{2}} x(t-\xi)\,\mathrm{d}\xi\right)''' + \frac{1}{4}\int_{2\pi}^{3\pi} x\left(t - \frac{\xi}{2}\right)\mathrm{d}\xi = 0$$

则 $a = 0$，$b = \frac{\pi}{2}$，$c = 2\pi$，$d = 3\pi$，$r(t) = 1$，$p(t, \xi) = \frac{1}{2}$，$\tau(t, \xi) = t - \xi$，$\sigma(t, \xi) = t - \frac{\xi}{2}$，$q(t, \xi) = \frac{1}{4}$，$\alpha = 1$，$\beta = 1$，$\lambda = 1$. 选取 $\rho(t) = 1$，显然(9) 式成立，且

$$Q(t) = \int_c^d q(t, \xi)\,\mathrm{d}\xi = \int_{2\pi}^{3\pi} \frac{1}{4}\mathrm{d}\xi = \frac{\pi}{4}$$

$$\sigma_1(t) = \sigma(t, 2\pi) = t - \pi$$

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\,\mathrm{d}t = \int_{t_0}^{\infty} \mathrm{d}s = \infty,$$

$$\int_a^b p(t, \xi)\,\mathrm{d}\xi = \int_0^{\frac{\pi}{2}} \frac{1}{2}\mathrm{d}\xi = \frac{\pi}{4} < 1$$

$$K(t) = \frac{\int_{t_2}^{t-\pi}\int_{t_1}^{s} \mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t} \mathrm{d}s} = \frac{(t - \pi - t_1)^2 - (t_2 - t_1)^2}{2(t - t_1)}$$

$$\limsup_{t\to\infty}\int_T^t \left(\delta\rho(s)K(s)Q(s) - \frac{1}{(\lambda+1)^{\lambda+1}}\frac{r(s)(\rho'(s))^{\lambda+1}}{\rho^{\lambda}(s)}\right)\mathrm{d}s =$$

$$\limsup_{t\to\infty}\int_T^t \delta(s - \pi)\frac{(s - \pi - t_1)^2 - (t_2 - t_1)^2}{2(s - t_1)}\mathrm{d}s = \infty$$

即(4) 式成立，定理 3 条件满足，可得例题中方程的解振动或$\lim\limits_{t\to\infty} x(t) = 0$. 事实上，可以验证 $x(t) = \cos t$ 就是该方程的振动解.

参考文献：

[1]　仉志余，王晓霞，俞元洪. 三阶半线性中立型分布时滞微分方程的振动性 [J]. 应用数学学报，2015，38(3)：450-459.

[2]　JIANG C M，JIANG Y，LI T X. Asymptotic Behavior of Third-Order Differential Equations with Nonpositive Neutral Coefficients and Distributed Deviating Arguments [J]. Advances in Difference Equations，2016，105：1-14.

[3]　林全文，俞元洪. 三阶半线性时滞微分方程的振动性和渐近性 [J]. 系统科学与数学，2015，35(2)：233-244.

[4]　屈　英. 三阶非线性中立型方程的 Philos 型振动定理 [J]. 数学的实践与认识，2011，41(7)：247-252.

[5]　罗李平，俞元洪. 三阶半线性中立型微分方程的振动结果 [J]. 系统科学与数学，2012，32(5)：571-579.

[6]　FU Y L，TIAN Y Z，JIANG C M，et al. On the Asymptotic Properties of Nonlinear Third-Order Neutral Delay Differential Equations with Distributed Deviating Arguments [J]. Journal of Function Spaces，2016，2016：1-5.

[7]　AKTAS M F，TIRYAKI A，ZAFER A. Oscillation Criteria for Third-Order Nonlinear Functional Differential Equations [J]. Applied Mathematics Letters，2010，23(7)：756-762.

[8]　BACULIKOVA B，DZURINA J. Oscillation of Third-Order Neutral Differential Dquations [J]. Mathematical and Computer Modelling，2010，52(1-2)：215-226.

[9]　CANDAN T. Asymptotic Properties of Solutions of Third-Order Nonlinear Neutral Dynamic Equations [J]. Advances in Difference Equations，2014，35：1-10.

[10]　SUN Y B，ZHAO Y G. Oscillation Criteria for Third-Order Nonlinear Neutral Differential Equations with Distributed

Deviating Arguments [J]. Applied Mathematics Letters, 2021, 25(10): 1514-1519.

[11] MOJSEJ I. Asymptotic Properties of Solutions of Third Order Nonlinear Differential Equations with Deviating Argument [J]. Nonlinear Analysis, 2008, 68(11): 3581-3591.

[12] LIU H D, MENG F W, LIU P C. Oscillation and Asymptotic Analysis on a New Generalized Emden-Fowler Equation [J]. Appied Mathematics and Computation, 2012, 219: 2739-2748.

[13] 曾云辉,罗李平,俞元洪. 中立型 Emden-Fowler 时滞微分方程的振动性 [J]. 数学物理学报, 2015, 35(4): 803-814.

[14] AKTAS M F, CAKMAK D, TIRYAKI A. On the Qualitative Behaviors of Solutions of Third-Order Nonlinear Functional Differential Equations [J]. Appied Mathematics Letters, 2011, 24: 1849-1855.

[15] CANDAN T. Oscillation Criteria and Asymptotic Properties of Solutions of Third-Order Nonlinear Neutral Differential Equations [J]. Mathematical Methods in the Applied Sciences, 2014, 38(7): 1379-1392.

[16] 魏 娟,朱朝生. 非局部非线性 Schrödinger 方程组解的渐近行为 [J]. 西南大学学报(自然科学版), 2019, 41(2): 60-63.

[17] 徐 茜. 空间异质环境下带交错扩散项的 Lotka-Volterra 模型分岔解的稳定性 [J]. 西南大学学报(自然科学版), 2018, 40(11): 35-40.

[18] 刘诗焕,朱先阳. 高维空间中阻尼 Boussinesq 方程初值问题的整体解 [J]. 西南师范大学学报(自然科学版), 2018, 43(9): 1-5.

[19] 赵阳洋,崔泽建. 一类带非局部源的反应扩散方程解的整体存在与爆破 [J]. 西南师范大学学报(自然科学版), 2019, 44(8): 34-38.

[20] 王开敏,李中平. 一类非线性伪抛物系统的全局解与非全局解 [J]. 贵州师范大学学报(自然科学版), 2018, 36(2): 59-63.

On Oscillation of Third-Order Neutral Differential Equations with Distributed Delay

LIU Jun[1], LIU Xi[2], ZHU Chun-yan[3], YU Yuan-hong[4]

1. Institute of Applied Mathematics, Qujing Normal University, Qujing Yunnan 655011, China;

2. School of Information Science and Engineering, Yunnan University, Kunming 650091, China;

3. College of Computer Science and Engineering, Qujing Normal University, Qujing Yunnan 655011, China;

4. Academy of Mathematics and systems Science, China Academy Science, Beijing 100190, China

Abstract: The objective of this paper is to study the oscillation of a class of third-order neutral differential equations with distributed delay. In this paper, not only the situation of $\beta \geqslant \alpha$, but also the situation of $\beta < \alpha$ has been studied. By means of the generalized Riccati transformation technique and special techniques, some new sufficient conditions for all solutions of the equation to be oscillatory or asymptotically convergent to zero have been obtained.

Key words: oscillation criterion; differential equation; generalized Riccati transformation; distributed delay

责任编辑 张 枸