DOI:10.13718/j.cnki.xsxb.2021.07.004

一类具混合边值条件的双参数奇摄动问题[①]

刘 燕

安徽师范大学皖江学院 电子工程系,安徽 芜湖 241000

摘要: 考虑一类带非线性混合边值条件的四阶微分方程的双参数奇摄动问题,在适当的条件下,运用合成展开法,构造在两个小参数相互关联的 3 种不同情形下双参数奇摄动问题的形式渐近解,利用微分不等式理论证明了在 3 种情形下原问题解的存在性和渐近解的一致有效性.

关 键 词: 奇摄动;非线性混合边值条件;双参数;微分不等式理论

中图分类号: O175.14 **文献标志码:** A **文章编号:** 1000-5471(2021)07-0026-13

非线性微分方程奇摄动问题在生态环境、大气物理、反应扩散、化学反应等自然科学领域中有很广泛的应用[1-3],因此非线性奇摄动方程的求解问题成为热门的研究课题.在自然界中,有些奇摄动问题涉及到双参数的边值问题,双参数奇摄动问题较单参数奇摄动问题情况复杂,对于双参数奇摄动问题,学者们也做了不少的工作[4-9].值得注意的是,这些工作主要集中在分离型的边界条件.受文献[4-7]的启发,本文运用合成展开法和微分不等式理论对一类具非线性混合边值条件的四阶非线性方程的双参数奇摄动问题解的结构作了较为全面的讨论,得到相应结果.考虑如下非线性混合边值条件的双参数奇摄动问题

$$\varepsilon y^{(4)} = f(t, y, y', y'', \mu y'''),\ a < t < c \tag{1}$$

$$y(b) = B_0 \tag{2}$$

$$y'(b) = B_1 \tag{3}$$

$$g(y''(a), y''(b), y''(c), y'''(a)) = 0 \tag{4}$$

$$h(y''(a), y''(b), y''(c), y'''(c)) = 0 \tag{5}$$

其中:$a < b < c$;B_0,B_1 为常数;ε 和 μ 是小的正参数.

本文总假设:

(H_1)退化问题

$$f(t, U, U', U'', 0) = 0$$

$$U(b) = B_0$$

$$U'(b) = B_1$$

在 $t \in [a, c]$ 上存在充分光滑的解 $U = U(t)$.

(H_2)函数 $f(t, y, y', y'', y''')$,$g(y_1, y_2, y_3, \rho)$,$h(y_1, y_2, y_3, \theta)$ 关于其变元在相应的区域内充分光滑,且各变量的各阶偏导都有界.$g_{y_2} \leqslant 0$,$g_{y_3} \leqslant 0$,$g_\rho < 0$,$h_{y_1} \leqslant 0$,$h_{y_2} \leqslant 0$,$h_\theta > 0$,且存在正常数 k,

① 收稿日期:2019-09-07
基金项目:安徽省高校自然科学研究项目(KJ2020A1190).
作者简介:刘 燕,讲师,硕士,主要从事奇异摄动理论研究.

$l_0, l_1, \delta_0, \delta_1, \delta_2$ 使得

$$f_{y'''} \geqslant k,\ |f_y| \leqslant \delta_1,\ |f_{y'}| \leqslant \delta_2$$

$$f_{y''} \geqslant \frac{\delta_1}{2}\max\{(a-b)^2,\ (c-b)^2\} + \delta_2\max\{b-a,\ c-b\} + \frac{\delta_0}{2}$$

$$g_{y_1} + g_{y_2} + g_{y_3} \geqslant l_0,\ h_{y_1} + h_{y_2} + h_{y_3} \geqslant l_1$$

（H_3）由

$$g(U''(a),\ U''(b),\ U''(c),\ U'''(a) + z_1) = 0$$

可求出 z_1；由

$$h(U''(a),\ U''(b),\ U''(c),\ U'''(c) - z_2) = 0$$

可求出 z_2.

现对小参数 ε 和 μ 讨论如下 3 种情形：

1）$\frac{\varepsilon}{\mu^2} \to 0$，$\mu \to 0$；

2）$\varepsilon = \mu^2$，$\mu \to 0$；

3）$\frac{\mu^2}{\varepsilon} \to 0$，$\varepsilon \to 0$.

1　预备知识

定义 1[10]　若 $\alpha(t), \beta(t) \in C^4[a,c]$，满足 $\alpha''(t) \leqslant \beta''(t)$，$t \in [a,c]$，且当 $t \in [a,c]$，y 介于 $\alpha(t)$，$\beta(t)$ 之间以及 y' 介于 $\alpha'(t)$，$\beta'(t)$ 之间时满足如下不等式：

$$\alpha^{(4)}(t) \geqslant f(t,\ y,\ y',\ \alpha''(t),\ \alpha'''(t))$$

$$\beta^{(4)}(t) \leqslant f(t,\ y,\ y',\ \beta''(t),\ \beta'''(t))$$

则分别称 $\beta(t)$，$\alpha(t)$ 为方程 $y^{(4)} = f(t,\ y,\ y',\ y'',\ y''')$ 的上下解.

定义 2[10]　若 $f(t,\ y,\ y',\ y'',\ y''')$ 在 $D = D_1 \cup D_2$ 上有定义且满足

$$|f(t,\ y,\ y',\ y'',\ y''')| \leqslant h(|y'''|),$$

其中 $h(s)$ 是定义在 $(0, +\infty)$ 上的正单调不减函数，满足

$$\int_{\lambda}^{+\infty} \frac{s\,\mathrm{d}s}{h(s)} > \max_{t\in[a,c]}\beta''(t) - \min_{t\in[a,c]}\alpha''(t)$$

其中 λ 为正常数且

$$\lambda(c-a) > \max\{|\beta''(c) - \alpha''(a)|,\ |\alpha''(c) - \beta''(a)|\}$$

则称 $f(t,\ y,\ y',\ y'',\ y''')$ 在 D 上关于 y''' 满足 Nagumo 条件，其中

$$D_1 = [a,\ b] \times [\alpha(t),\ \beta(t)] \times [\beta'(t),\ \alpha'(t)] \times [\alpha''(t),\ \beta''(t)] \times \mathbb{R}$$

$$D_2 = [b,\ c] \times [\alpha(t),\ \beta(t)] \times [\alpha'(t),\ \beta'(t)] \times [\alpha''(t),\ \beta''(t)] \times \mathbb{R}$$

引理 1[10]　若边值问题

$$y^{(4)} = f(t,\ y,\ y',\ y'',\ y'''),\ a < t < c \tag{6}$$

$$y(b) = B_0 \tag{7}$$

$$y'(b) = B_1 \tag{8}$$

$$g(y''(a),\ y''(b),\ y''(c),\ y'''(a)) = 0 \tag{9}$$

$$h(y''(a),\ y''(b),\ y''(c),\ y'''(c)) = 0 \tag{10}$$

其中 $a < b < c$，满足如下条件：

(i) 具有下解 $\alpha(t)$ 与上解 $\beta(t)$，使得

$$\alpha(b)=\beta(b)=B_0$$

$$\alpha'(b)=\beta'(b)=B_1$$

$$g(\alpha''(a),\ \alpha''(b),\ \alpha''(c),\ \alpha'''(a))\leqslant 0\leqslant g(\beta''(a),\ \beta''(b),\ \beta''(c),\ \beta'''(a))$$

$$h(\alpha''(a),\ \alpha''(b),\ \alpha''(c),\ \alpha'''(c))\leqslant 0\leqslant h(\beta''(a),\ \beta''(b),\ \beta''(c),\ \beta'''(c))$$

(ii) $f(t,\ y,\ y',\ y'',\ y''')$ 在 D 上连续且关于 y''' 满足 Nagumo 条件.

(iii) $g(y_1,\ y_2,\ y_3,\ \rho)$ 在 $\overline{D}=[\alpha''(a),\ \beta''(a)]\times[\alpha''(b),\ \beta''(b)]\times[\alpha''(c),\ \beta''(c)]\times[-N,\ N]$ 上连续且关于 y_2,y_3,ρ 单调不增;$h(y_1,\ y_2,\ y_3,\ \theta)$ 在 $\overline{D}$ 上连续且关于 y_1,y_2 单调不增,关于 θ 单调不减,则边值(6)-(10) 存在解 $y(t)\in C^4[a,\ c]$,满足

$$\alpha''(t)\leqslant y''(t)\leqslant\beta''(t),\ |\ y'''(t)\ |\leqslant N,\ t\in[a,\ c]$$

注 1 在上述引理条件下,易证

$$\alpha(t)\leqslant y(t)\leqslant\beta(t),\ t\in[a,\ c]$$

2 主要结果

情形 1 当 $\frac{\varepsilon}{\mu^2}\to 0$,$\mu\to 0$ 时形式渐近解的构造.

令 $\varepsilon_1=\mu$,$\varepsilon_2=\frac{\varepsilon}{\mu^2}$,则方程(1) 转化为:

$$\varepsilon_1^2\varepsilon_2 y^{(4)}=f(t,\ y,\ y',\ y'',\ \varepsilon_1 y''')$$

设外部解的形式渐近式为

$$Y(t,\ \varepsilon_1,\ \varepsilon_2)\sim\sum_{i,j=0}^{\infty}Y_{i,j}(t)\varepsilon_1^i\varepsilon_2^j$$

则可得

$$f(t,\ Y_{0,0},\ Y'_{0,0},\ Y''_{0,0},\ 0)=0 \tag{11}$$

$$Y_{0,0}(b)=B_0 \tag{12}$$

$$Y'_{0,0}(b)=B_1 \tag{13}$$

以及

$$f_y(t,\ Y_{0,0},\ Y'_{0,0},\ Y''_{0,0},\ 0)Y_{i,j}+f_{y'}(t,\ Y_{0,0},\ Y'_{0,0},\ Y''_{0,0},\ 0)Y'_{i,j}+$$
$$f_{y''}(t,\ Y_{0,0},\ Y'_{0,0},\ Y''_{0,0},\ 0)Y''_{i,j}=F_{i,j}(t),\ i+j\geqslant 1 \tag{14}$$

$$Y_{i,j}(b)=0,\ i+j\geqslant 1 \tag{15}$$

$$Y'_{i,j}(b)=0,\ i+j\geqslant 1 \tag{16}$$

其中 $F_{i,j}(t)$ 是由 $Y_{s,q}$,$Y'_{s,q}$,$Y''_{s,q}$,$Y'''_{s,q}$,$Y^{(4)}_{s,q}(s+q<i+j)$ 依次确定的函数,特别地 $F_{0,1}(t)=0$,由(11)-(16) 式结合假设$[H_1]$可依次确定 $Y_{i,j}(t)$.

令

$$y(t,\ \varepsilon_1,\ \varepsilon_2)=Y(t,\ \varepsilon_1,\ \varepsilon_2)+\varepsilon_1^3\hat{\mu}(\tau_1,\ \varepsilon_1,\ \varepsilon_2)$$

其中 $\tau_1=\frac{t-a}{\varepsilon_1}$ 为伸长变量,且

$$\hat{\mu}(\tau_1,\ \varepsilon_1,\ \varepsilon_2)\sim\sum_{i,\ j=0}^{\infty}\hat{\mu}_{i,\ j}(\tau_1)\varepsilon_1^i\varepsilon_2^j$$

具有性质

$$\lim_{\tau_1\to+\infty}\hat{\mu}_{i,\ j}(\tau_1,\ \varepsilon_1,\ \varepsilon_2)=\lim_{\tau_1\to+\infty}\frac{\mathrm{d}\hat{\mu}_{i,\ j}}{\mathrm{d}\tau_1}(\tau_1,\ \varepsilon_1,\ \varepsilon_2)=0 \tag{17}$$

则有

$$f_{y''}(a, Y_{0,0}(a), Y'_{0,0}(a), Y''_{0,0}(a), 0)\frac{d^2\hat{\mu}_{0,0}}{d\tau_1^2}+f_{y'''}(a, Y_{0,0}(a), Y'_{0,0}(a), Y''_{0,0}(a), 0)\frac{d^3\hat{\mu}_{0,0}}{d\tau_1^3}=0 \tag{18}$$

$$f_{y''}(a, Y_{0,0}(a), Y'_{0,0}(a), Y''_{0,0}(a), 0)\frac{d^2\hat{\mu}_{0,0}}{d\tau_1^2}+$$
$$f_{y'''}(a, Y_{0,0}(a), Y'_{0,0}(a), Y''_{0,0}(a), 0)\frac{d^3\hat{\mu}_{0,0}}{d\tau_1^3}=\widetilde{F}_{i,j}(\tau_1),\ i+j\geqslant 1 \tag{19}$$

其中 $\widetilde{F}_{i,j}(\tau_1)$ 是 τ_1，$\mu_{s,q}(s+q<i+j)$ 及其各阶导数的多项式函数.

令

$$y(t, \varepsilon_1, \varepsilon_2)=Y(t, \varepsilon_1, \varepsilon_2)+\varepsilon_1^3\varepsilon_2^3\nu(\tau_2, \varepsilon_1, \varepsilon_2)$$

其中 $\tau_2=\dfrac{c-t}{\varepsilon_1\varepsilon_2}$ 为伸长变量，且

$$\nu(\tau_2, \varepsilon_1, \varepsilon_2)\sim\sum_{i,j=0}^{\infty}\nu_{i,j}(\tau_2)\varepsilon_1^i\varepsilon_2^j$$

其具有性质

$$\lim_{\tau_2\to+\infty}\nu_{i,j}(\tau_2, \varepsilon_1, \varepsilon_2)=\lim_{\tau_2\to+\infty}\frac{d\nu_{i,j}}{d\tau_2}(\tau_2, \varepsilon_1, \varepsilon_2)=\lim_{\tau_2\to+\infty}\frac{d^2\nu_{i,j}}{d\tau_2^2}(\tau_2, \varepsilon_1, \varepsilon_2)=0 \tag{20}$$

则有

$$\frac{d^4\nu_{0,0}}{d\tau_2^4}=-f_{y'''}(c, Y_{0,0}(c), Y'_{0,0}(c), Y''_{0,0}(c), 0)\frac{d^3\nu_{0,0}}{d\tau_2^3} \tag{21}$$

$$\frac{d^4\nu_{i,j}}{d\tau_2^4}=-f_{y'''}(c, Y_{0,0}(c), Y'_{0,0}(c), Y''_{0,0}(c), 0)\frac{d^3\nu_{i,j}}{d\tau_2^3}+\overline{F}_{i,j}(\tau_2),\ i+j\geqslant 1 \tag{22}$$

其中 $\overline{F}_{i,j}(\tau_2)$ 是关于 τ_2，$\nu_{s,q}(s+q<i+j)$ 及其各阶导数的多项式函数.

为了确定 $\hat{\mu}_{i,j}(\tau_1)$，$\nu_{i,j}(\tau_2)$ 所确定的定解条件，将

$$y(t, \varepsilon_1, \varepsilon_2)=Y(t, \varepsilon_1, \varepsilon_2)+\varepsilon_1^3\hat{\mu}(\tau_1, \varepsilon_1, \varepsilon_2)+\varepsilon_1^3\varepsilon_2^3\nu(\tau_2, \varepsilon_1, \varepsilon_2)$$

代入(4)和(5)式，可得

$$g\left(Y''_{0,0}(a), Y''_{0,0}(b), Y''_{0,0}(c), Y'''_{0,0}(a)+\left.\frac{d^3\hat{\mu}_{0,0}}{d\tau_1^3}\right|_{\tau_1=0}\right)=0 \tag{23}$$

$$h\left(Y''_{0,0}(a), Y''_{0,0}(b), Y''_{0,0}(c), Y'''_{0,0}(c)-\left.\frac{d^3\nu_{0,0}}{d\tau_2^3}\right|_{\tau_2=0}\right)=0 \tag{24}$$

$$g_\rho\left(Y''_{0,0}(a), Y''_{0,0}(b), Y''_{0,0}(c), Y'''_{0,0}(a)+\left.\frac{d^3\hat{\mu}_{0,0}}{d\tau_1^3}\right|_{\tau_1=0}\right)\left.\frac{d^3\hat{\mu}_{i,j}}{d\tau_1^3}\right|_{\tau_1=0}=G_{i,j},\ i+j\geqslant 1 \tag{25}$$

$$h_\theta\left(Y''_{0,0}(a), Y''_{0,0}(b), Y''_{0,0}(c), Y'''_{0,0}(c)-\left.\frac{d^3\nu_{0,0}}{d\tau_2^3}\right|_{\tau_2=0}\right)\left.\frac{d^3\nu_{i,j}}{d\tau_2^3}\right|_{\tau_2=0}=H_{i,j},\ i+j\geqslant 1 \tag{26}$$

其中 $G_{i,j}$，$H_{i,j}$ 是确定的常数. 由假设(H_3)及(23)－(24)式可求出 $\left.\dfrac{d^3\hat{\mu}_{0,0}}{d\tau_1^3}\right|_{\tau_1=0}$，记为 U_0；$\left.\dfrac{d^3\nu_{0,0}}{d\tau_2^3}\right|_{\tau_2=0}$，记为 V_0. 再结合(17)－(22)，(25)，(26)式以及假设(H_2)可求出 $\hat{\mu}_{i,j}(\tau_1)$，$\nu_{i,j}(\tau_2)$ 其中

$$\hat{\mu}_{0,0}(\tau_1)=\frac{U_0}{\lambda_1^3}e^{\lambda_1\tau_1}$$

$$\nu_{0,0}(\tau_2)=\frac{V_0}{\lambda_2^3}e^{\lambda_2\tau_2}$$

λ_1 为(18)式的特征方程的一个负根，λ_2 为(21)式的特征方程的一个负根. 由 $\hat{\mu}_{0,0}(\tau_1)$，$\nu_{0,0}(\tau_2)$ 及 $\widetilde{F}_{i,j}(\tau_1)$，$\overline{F_{i,j}}(\tau_2)$ 的构造，可知 $\hat{\mu}_{i,j}(\tau_1)$，$\nu_{i,j}(\tau_2)$ 都具有指数型衰减的特征. 引进光滑函数 $\psi(t) \in C^{\infty}$，使得

$$\psi(t)=\begin{cases}1,\ t\in[0,\sigma],\\ p(t),\ t\in[\sigma,2\sigma]\\ 0,\ t\in[2\sigma,c-a],\end{cases}$$

其中 $p(t)$ 是个多项式函数，常数 σ 满足

$$0<\sigma<\frac{c-a}{2}$$

令

$$y_m(t,\varepsilon_1,\varepsilon_2)=\sum_{i,j=0}^{m}Y_{i,j}(t)\varepsilon_1^i\varepsilon_2^j+\psi(t-a)\varepsilon_1^3\sum_{i,j=0}^{m}\hat{\mu}_{i,j}\left(\frac{t-a}{\varepsilon_1}\right)\varepsilon_1^i\varepsilon_2^j+\psi(c-t)\varepsilon_1^3\varepsilon_2^3\sum_{i,j=0}^{m}\nu_{i,j}\left(\frac{c-t}{\varepsilon_1\varepsilon_2}\right)\varepsilon_1^i\varepsilon_2^j \tag{27}$$

得到问题(1)-(5)的 m 阶形式渐近解，其中 $\varepsilon_1=\mu$，$\varepsilon_2=\frac{\varepsilon}{\mu^2}$ 且 $\frac{\varepsilon}{\mu^2}\to 0$，$\mu\to 0$. 下面证明问题(1)-(5)解的存在性及 m 阶形式渐近展开式(27)的一致有效性.

定理 1 若假设[H_1]~[H_3]成立，且存在定义在$(0,+\infty)$上单调不减的正值函数 $h(s)$，满足

$$\int_1^{+\infty}\frac{s\mathrm{d}s}{h(s)}=+\infty$$

使得对$[a,c]\times\mathbb{R}^3$ 的紧子集中的(t,y,y',y'')及 $y'''\in\mathbb{R}$ 有

$$|f(t,y,y',y'',y''')|\leqslant h(|y'''|),$$

则对正的小参数 ε 和 μ，当 $\frac{\varepsilon}{\mu^2}\to 0(\mu\to 0)$ 时两参数问题(1)-(5)有解

$$y=y(t,\varepsilon_1,\varepsilon_2)\in C^4[a,c]$$

满足

$$y^{(n)}(t,\varepsilon_1,\varepsilon_2)=y_m^{(n)}(t,\varepsilon_1,\varepsilon_2)+O(d_0^{m+1}),\ n=0,1,2,\ t\in[a,c]$$

其中 $y_m(t,\varepsilon_1,\varepsilon_2)$ 由(27)式给出，$d_0=\max\{\varepsilon_1,\varepsilon_2\}$，$\varepsilon_1=\mu$，$\varepsilon_2=\frac{\varepsilon}{\mu^2}$.

证 构造界定函数

$$\alpha(t,\varepsilon_1,\varepsilon_2)=y_m(t,\varepsilon_1,\varepsilon_2)-r(t-b)^2d_0^{m+1}$$

$$\beta(t,\varepsilon_1,\varepsilon_2)=y_m(t,\varepsilon_1,\varepsilon_2)+r(t-b)^2d_0^{m+1}$$

显然

$$\alpha''(t,\varepsilon_1,\varepsilon_2)\leqslant\beta''(t,\varepsilon_1,\varepsilon_2),\ t\in[a,c]$$

$$\beta'(t,\varepsilon_1,\varepsilon_2)\leqslant\alpha'(t,\varepsilon_1,\varepsilon_2),\ t\in[a,b]$$

$$\alpha'(t,\varepsilon_1,\varepsilon_2)\leqslant\beta'(t,\varepsilon_1,\varepsilon_2),\ t\in[b,c]$$

$$\alpha(b,\varepsilon_1,\varepsilon_2)=B_0=\beta(b,\varepsilon_1,\varepsilon_2)$$

$$\alpha'(b,\varepsilon_1,\varepsilon_2)=B_1=\beta'(b,\varepsilon_1,\varepsilon_2)$$

另外，由微分中值定理，存在正常数 M_1，M_2 使

$$\begin{aligned}&g(\alpha''(a,\varepsilon_1,\varepsilon_2),\alpha''(b,\varepsilon_1,\varepsilon_2),\alpha''(c,\varepsilon_1,\varepsilon_2),\alpha'''(a,\varepsilon_1,\varepsilon_2))\leqslant\\&g(y''_m(a,\varepsilon_1,\varepsilon_2),y''_m(b,\varepsilon_1,\varepsilon_2),y''_m(c,\varepsilon_1,\varepsilon_2),y'''_m(a,\varepsilon_1,\varepsilon_2))-2l_0rd_0^{m+1}\leqslant\\&g\left(Y''_{0,0}(a),Y''_{0,0}(b),Y''_{0,0}(c),Y'''_{0,0}(a)+\left.\frac{\mathrm{d}^3\hat{\mu}_{0,0}}{\mathrm{d}\tau_1^3}\right|_{\tau_1=0}\right)+\end{aligned}$$

$$\sum_{i+j=1}^{m}(g_\rho(Y''_{0,0}(a),Y''_{0,0}(b),Y''_{0,0}(c),Y'''_{0,0}(a)+\frac{d^3\hat{\mu}_{0,0}}{d\tau_1^3}\Big|_{\tau_1=0})\frac{d^3\hat{\mu}_{i,j}}{d\tau_1^3}\Big|_{\tau_1=0}-G_{i,j})\varepsilon_1^i\varepsilon_2^j+M_1d_0^{m+1}-2l_0rd_0^{m+1}=$$
$$(M_1-2l_0r)d_0^{m+1}$$

以及

$$h(\alpha''(a,\varepsilon_1,\varepsilon_2),\alpha''(b,\varepsilon_1,\varepsilon_2),\alpha''(c,\varepsilon_1,\varepsilon_2),\alpha'''(c,\varepsilon_1,\varepsilon_2))\leqslant$$
$$h(y''_m(a,\varepsilon_1,\varepsilon_2),y''_m(b,\varepsilon_1,\varepsilon_2),y''_m(c,\varepsilon_1,\varepsilon_2),y'''_m(c,\varepsilon_1,\varepsilon_2))-2l_1rd_0^{m+1}\leqslant$$
$$h(Y''_{0,0}(a),Y''_{0,0}(b),Y''_{0,0}(c),Y'''_{0,0}(c)-\frac{d^3\nu_{0,0}}{d\tau_2^3}\Big|_{\tau_2=0})+$$
$$\sum_{i+j=1}^{m}(h_\theta(Y''_{0,0}(a),Y''_{0,0}(b),Y''_{0,0}(c),Y'''_{0,0}(c)-\frac{d^3\nu_{0,0}}{d\tau_2^3}\Big|_{\tau_2=0})\frac{d^3\nu_{i,j}}{d\tau_2^3}\Big|_{\tau_2=0}-H_{i,j})\varepsilon_1^i\varepsilon_2^j+M_2d_0^{m+1}-2l_1rd_0^{m+1}=$$
$$(M_2-2l_1r)d_0^{m+1}$$

只要 $r\geqslant\max\left\{\frac{M_1}{2l_0},\frac{M_2}{2l_1}\right\}$，就有

$$g(\alpha''(a,\varepsilon_1,\varepsilon_2),\alpha''(b,\varepsilon_1,\varepsilon_2),\alpha''(c,\varepsilon_1,\varepsilon_2),\alpha'''(a,\varepsilon_1,\varepsilon_2))\leqslant 0$$
$$h(\alpha''(a,\varepsilon_1,\varepsilon_2),\alpha''(b,\varepsilon_1,\varepsilon_2),\alpha''(c,\varepsilon_1,\varepsilon_2),\alpha'''(c,\varepsilon_1,\varepsilon_2))\leqslant 0$$

最后，对任意的 s_1 介于 $\alpha(t,\varepsilon_1,\varepsilon_2)$，$\beta(t,\varepsilon_1,\varepsilon_2)$ 之间以及任意的 s_2 介于 $\alpha'(t,\varepsilon_1,\varepsilon_2)$，$\beta'(t,\varepsilon_1,\varepsilon_2)$ 之间有

$$\varepsilon\alpha^{(4)}-f(t,s_1,s_2,\alpha'',\mu\alpha''')=$$
$$\varepsilon y_m^{(4)}-f(t,y_m,y'_m,y''_m,\mu y'''_m)-f_y(t,\theta_0,\theta_1,\theta_2,\mu y'''_m)(s_1-y_m)-f_{y'}(t,\theta_0,\theta_1,\theta_2,\mu y'''_m)(s_2-y'_m)+$$
$$2f_{y''}(t,\theta_0,\theta_1,\theta_2,\mu y'''_m)rd_0^{m+1}\geqslant$$
$$\varepsilon y_m^{(4)}-f(t,y_m,y'_m,y''_m,\mu y'''_m)+\delta_0rd_0^{m+1}$$

其中 θ_0 介于 s_1 与 y_m 之间，θ_1 介于 s_2 与 y'_m 之间，θ_2 介于 α'' 与 y''_m 之间.

当 $x\in[a,a+\sigma]$ 时，由左边界层构造知，存在正常数 M_3，M_4，使得

$$\varepsilon y_m^{(4)}-f(t,y_m,y'_m,y''_m,\mu y'''_m)+\delta_0rd_0^{m+1}\geqslant$$
$$f(t,Y_{0,0},Y'_{0,0},Y''_{0,0},0)+\sum_{i+j=1}^{m}(f_y(t,Y_{0,0},Y'_{0,0},Y''_{0,0},0)Y_{i,j}+f_{y'}(t,Y_{0,0},Y'_{0,0},Y''_{0,0},0)Y'_{i,j}+$$
$$f_{y''}(t,Y_{0,0},Y'_{0,0},Y''_{0,0},0)Y''_{i,j}-F_{i,j}(t))\varepsilon_1^i\varepsilon_2^j-M_3d_0^{m+1}+(f_{y''}(a,Y_{0,0}(a),Y'_{0,0}(a),Y''_{0,0}(a),0)\frac{d^2\hat{\mu}_{0,0}}{d\tau_1^2}+$$
$$f_{y'''}(a,Y_{0,0}(a),Y'_{0,0}(a),Y''_{0,0}(a),0)\frac{d^3\hat{\mu}_{0,0}}{d\tau_1^3})\varepsilon_1^1\varepsilon_2^0+\sum_{i+j=1}^{m}(f_{y''}(a,Y_{0,0}(a),Y'_{0,0}(a),Y''_{0,0}(a),0)\frac{d^2\hat{\mu}_{i,j}}{d\tau_1^2}+$$
$$f_{y'''}(a,Y_{0,0}(a),Y'_{0,0}(a),Y''_{0,0}(a),0)\frac{d^3\hat{\mu}_{i,j}}{d\tau_1^3}-\widetilde{F}_{i,j}(\tau_1))\varepsilon_1^{i+1}\varepsilon_2^j-M_4d_0^{m+1}+\delta_0rd_0^{m+1}=$$
$$(\delta_0r-M_3-M_4)d_0^{m+1}$$

类似地，当 $x\in[c-\sigma,c]$ 时，由右边界层构造知，存在正常数 M_5，使得

$$\varepsilon y_m^{(4)}-f(t,y_m,y'_m,y''_m,\mu y'''_m)+\delta_0rd_0^{m+1}\geqslant(\delta_0r-M_3-M_5)d_0^{m+1}$$

当 $x\in[a+\sigma,c-\sigma]$ 时，存在正常数 M_6，使得

$$\varepsilon y_m^{(4)}-f(t,y_m,y'_m,y''_m,\mu y'''_m)+\delta_0rd_0^{m+1}\geqslant$$
$$-M_3d_0^{m+1}+\varepsilon_1^2\varepsilon_2[\psi(t-a)\varepsilon_1^3\sum_{i,j=0}^{m}\hat{\mu}_{i,j}\left(\frac{t-a}{\varepsilon_1}\right)\varepsilon_1^i\varepsilon_2^j+\psi(c-t)\varepsilon_1^3\varepsilon_2^3\sum_{i,j=0}^{m}\nu_{i,j}\left(\frac{c-t}{\varepsilon_1\varepsilon_1}\right)\varepsilon_1^i\varepsilon_2^j]^{(4)}+$$

$$f_y(t,\gamma_0,\gamma_1,\gamma_2,\gamma_3)(\sum_{i,j=0}^{m}Y_{i,j}(t)\varepsilon_1^i\varepsilon_2^j-y_m)+f_{y'}(t,\gamma_0,\gamma_1,\gamma_2,\gamma_3)(\sum_{i,j=0}^{m}Y'_{i,j}(t)\varepsilon_1^i\varepsilon_2^j-y'_m)+$$

$$f_{y''}(t,\gamma_0,\gamma_1,\gamma_2,\gamma_3)(\sum_{i,j=0}^{m}Y''_{i,j}(t)\varepsilon_1^i\varepsilon_2^j-y''_m)+f_{y'''}(t,\gamma_0,\gamma_1,\gamma_2,\gamma_3)(\varepsilon_1\sum_{i,j=0}^{m}Y'''_{i,j}(t)\varepsilon_1^i\varepsilon_2^j-\varepsilon_1 y'''_m)+$$

$$\delta_0 rd_0^{m+1}\geqslant -M_3d_0^{m+1}-M_6d_0^{m+1}+\delta_0rd_0^{m+1}=(\delta_0r-M_3-M_6)d_0^{m+1}$$

其中 γ_k 在 $\sum_{i,j=0}^{m}Y_{i,j}^{(k)}(t)\varepsilon_1^i\varepsilon_2^j$ 与 $y_m^{(k)}(k=0,1,2)$ 之间，γ_3 在 $\varepsilon_1\sum_{i,j=0}^{m}Y'''_{i,j}(t)\varepsilon_1^i\varepsilon_2^j$ 与 $\varepsilon_1 y'''_m$ 之间.

只要取 $r\geqslant\max\left\{\frac{M_3+M_4}{\delta_0},\frac{M_3+M_5}{\delta_0},\frac{M_3+M_6}{\delta_0}\right\}$，就有

$$\varepsilon\alpha^{(4)}\geqslant f(t,s_1,s_2,\alpha'',\mu\alpha'''),\ t\in[a,c]$$

同理，只要 r 充分大，就有

$$\varepsilon\beta^{(4)}\leqslant f(t,s_1,s_2,\beta'',\mu\beta'''),\ t\in[a,c]$$

故由引理 1 可知，边值问题(1)－(5)有解 $y(t,\varepsilon_1,\varepsilon_2)\in C^4[a,c]$，满足

$$\beta'(t,\varepsilon_1,\varepsilon_2)\leqslant y'(t,\varepsilon_1,\varepsilon_2)\leqslant\alpha'(t,\varepsilon_1,\varepsilon_2),\ t\in[a,b]$$
$$\alpha'(t,\varepsilon_1,\varepsilon_2)\leqslant y'(t,\varepsilon_1,\varepsilon_2)\leqslant\beta'(t,\varepsilon_1,\varepsilon_2),\ t\in[b,c]$$
$$\alpha''(t,\varepsilon_1,\varepsilon_2)\leqslant y''(t,\varepsilon_1,\varepsilon_2)\leqslant\beta''(t,\varepsilon_1,\varepsilon_2),\ t\in[a,c]$$

由注 1 有

$$\alpha(t,\varepsilon_1,\varepsilon_2)\leqslant y(t,\varepsilon_1,\varepsilon_2)\leqslant\beta(t,\varepsilon_1,\varepsilon_2),\ t\in[a,c]$$

定理 1 证毕.

当$\frac{\varepsilon}{\mu^2}(\mu\to0)$时，问题(1)－(5)具有形如(27)式的渐近展开式，并在 $x=a$ 处和 $x=c$ 处附近各有一个薄层，其薄层宽度分别为 $O(\varepsilon_1)$ 和 $O(\varepsilon_1\varepsilon_2)$，因为 $\varepsilon_1>\varepsilon_1\varepsilon_2$，所以在 $x=c$ 处的薄层比在 $x=a$ 处的薄层更薄.

情形 2 当 $\varepsilon=\mu^2$，$\mu\to0$ 时形式渐近解的构造.

令 $\varepsilon_1=\mu$，则方程(1)转化为：

$$\varepsilon_1^2y^{(4)}=f(t,y,y',y'',\varepsilon_1y'''),\ a<t<c \tag{28}$$

用与情形 1 相同的正则摄动方法可得方程(28)对应的外部解：

$$Y(t,\varepsilon_1)\sim\sum_{j=0}^{\infty}Y_j(t)\varepsilon_1^j$$

令

$$y(t,\varepsilon_1)=Y(t,\varepsilon_1)+\varepsilon_1^3\tilde{\mu}(\tau_3,\varepsilon_1)$$

其中 $\tau_3=\frac{t-a}{\varepsilon_1}$ 为伸长变量，且

$$\tilde{\mu}(\tau_3,\varepsilon_1)\sim\sum_{j=0}^{\infty}\tilde{\mu}_j(\tau_3)\varepsilon_1^j$$

其具有性质

$$\lim_{\tau_3\to+\infty}\tilde{\mu}_j(\tau_3,\varepsilon_1)=\lim_{\tau_3\to+\infty}\frac{\mathrm{d}\tilde{\mu}_j}{\mathrm{d}\tau_3}(\tau_3,\varepsilon_1)=\lim_{\tau_3\to+\infty}\frac{\mathrm{d}^2\tilde{\mu}_j}{\mathrm{d}\tau_3^2}(\tau_3,\varepsilon_1)=0 \tag{29}$$

则有

$$\frac{\mathrm{d}^4\tilde{\mu}_0}{\mathrm{d}\tau_3^4}=f_{y''}(a,Y_0(a),Y'_0(a),Y''_0(a),0)\frac{\mathrm{d}^2\tilde{\mu}_0}{\mathrm{d}\tau_3^2}+f_{y'''}(a,Y_0(a),Y'_0(a),Y''_0(a),0)\frac{\mathrm{d}^3\tilde{\mu}_0}{\mathrm{d}\tau_3^3} \tag{30}$$

$$\frac{\mathrm{d}^4\tilde{\mu}_j}{\mathrm{d}\tau_3^4}=f_{y''}(a,Y_0(a),Y'_0(a),Y''_0(a),0)\frac{\mathrm{d}^2\tilde{\mu}_j}{\mathrm{d}\tau_3^2}+f_{y'''}(a,Y_0(a),Y'_0(a),Y''_0(a),0)\frac{\mathrm{d}^3\tilde{\mu}_j}{\mathrm{d}\tau_3^3}+\tilde{F}_{j-1}(\tau_3),\ j\geqslant1 \tag{31}$$

其中 $\widetilde{F}_{j-1}(j\geqslant 1)$ 是关于 $\tau_3,\tilde{\mu}_k(k\leqslant j-1)$ 及其各阶导数的多项式函数.

令

$$y(t,\varepsilon_1)=Y(t,\varepsilon_1)+\varepsilon_1^3\tilde{\nu}(\tau_4,\varepsilon_1)$$

其中 $\tau_4=\dfrac{c-t}{\varepsilon_1}$ 为伸长变量，且

$$\tilde{\nu}(\tau_4,\varepsilon_1)\sim\sum_{j=0}^{\infty}\tilde{\nu}_j(\tau_4)\varepsilon_1^j$$

其具有性质

$$\lim_{\tau_4\to+\infty}\tilde{\nu}_j(\tau_4,\varepsilon_1)=\lim_{\tau_4\to+\infty}\frac{d\tilde{\nu}_j}{d\tau_4}(\tau_4,\varepsilon_1)=\lim_{\tau_3\to+\infty}\frac{d^2\tilde{\nu}_j}{d\tau_4^2}(\tau_4,\varepsilon_1)=0 \tag{32}$$

则有

$$\frac{d^4\tilde{\nu}_0}{d\tau_4^4}=f_{y''}(c,Y_0(c),Y'_0(c),Y''_0(c),0)\frac{d^2\tilde{\nu}_0}{d\tau_4^2}-f_{y'''}(c,Y_0(c),Y'_0(c),Y''_0(c),0)\frac{d^3\tilde{\nu}_0}{d\tau_4^3} \tag{33}$$

$$\frac{d^4\tilde{\nu}_j}{d\tau_4^4}=f_{y''}(c,Y_0(c),Y'_0(c),Y''_0(c),0)\frac{d^2\tilde{\nu}_j}{d\tau_4^2}-f_{y'''}(c,Y_0(c),Y'_0(c),Y''_0(c),0)\frac{d^3\tilde{\nu}_j}{d\tau_4^3}+\bar{F}_{j-1}(\tau_4),\ j\geqslant 1 \tag{34}$$

其中 $\bar{F}_{j-1}(j\geqslant 1)$ 是关于 $\tau_4,\tilde{\nu}_k(k\leqslant j-1)$ 及其各阶导数的多项式函数.

为了确定 $\tilde{\mu}_j(\tau_3),\tilde{\nu}_j(\tau_4)$ 所满足的定解条件，令

$$y(t,\varepsilon_1)=Y(t,\varepsilon_1)+\varepsilon_1^3\tilde{\mu}(\tau_3,\varepsilon_1)+\varepsilon_1^3\tilde{\nu}(\tau_4,\varepsilon_1)$$

则有

$$g\Big(Y''_0(a),Y''_0(b),Y''_0(c),Y'''_0(a)+\frac{d^3\tilde{\mu}_0}{d\tau_3^3}\Big|_{\tau_3=0}\Big)=0 \tag{35}$$

$$h\Big(Y''_0(a),Y''_0(b),Y''_0(c),Y'''_0(c)-\frac{d^3\tilde{\nu}_0}{d\tau_4^3}\Big|_{\tau_4=0}\Big)=0 \tag{36}$$

$$g_\rho\Big(Y''_0(a),Y''_0(b),Y''_0(c),Y'''_0(a)+\frac{d^3\tilde{\mu}_0}{d\tau_3^3}\Big|_{\tau_3=0}\Big)\frac{d^3\tilde{\mu}_j}{d\tau_3^3}\Big|_{\tau_3=0}=\widetilde{G}_{j-1},\ j\geqslant 1 \tag{37}$$

$$h_\theta\Big(Y''_0(a),Y''_0(b),Y''_0(c),Y'''_0(c)-\frac{d^3\tilde{\nu}_0}{d\tau_4^3}\Big|_{\tau_4=0}\Big)\frac{d^3\tilde{\nu}_j}{d\tau_4^3}\Big|_{\tau_4=0}=\widetilde{H}_{j-1},\ j\geqslant 1 \tag{38}$$

其中 $\widetilde{G}_{j-1},\widetilde{H}_{j-1}$ 是依次确定的常数，由假设(H_3)及(35)－(36)式可求出$\dfrac{d^3\tilde{\mu}_0}{d\tau_3^3}\Big|_{\tau_3=0}$ 记为 U_1，$\dfrac{d^3\tilde{\nu}_0}{d\tau_4^3}\Big|_{\tau_4=0}$ 记为 V_1，再结合(29)－(34),(37),(38)式以及假设(H_2)可求出 $\tilde{\mu}_j(\tau_3),\tilde{\nu}_j(\tau_4)$，其中

$$\tilde{\mu}_0(\tau_3)=\frac{U_1}{\lambda_3^3}e^{\lambda_3\tau_3}$$

$$\tilde{\nu}_0(\tau_4)=\frac{V_1}{\lambda_4^3}e^{\lambda_4\tau_4}$$

λ_3 为(30)式的特征方程的一个负根，λ_4 为(33)式的特征方程的一个负根. 由 $\tilde{\mu}_0(\tau_3),\tilde{\nu}_0(\tau_4)$ 及 $\widetilde{F}_j(\tau_3)$，$\bar{F}_j(\tau_4)$ 的构造，可知 $\tilde{\mu}_j(\tau_3),\tilde{\nu}_j(\tau_4)$ 都具有指数型衰减的特征.

令

$$y_m(t,\varepsilon_1)=\sum_{j=0}^{m}Y_j(t)\varepsilon_1^j+\psi(t-a)\varepsilon_1^3\sum_{j=0}^{m}\tilde{\mu}_j\Big(\frac{t-a}{\varepsilon_1}\Big)\varepsilon_1^j+\psi(c-t)\varepsilon_1^3\sum_{j=0}^{m}\tilde{\nu}_j\Big(\frac{c-t}{\varepsilon_1}\Big)\varepsilon_1^j \tag{39}$$

得到问题(1)－(5)的 m 阶形式渐近解，其中 $\varepsilon_1=\mu$ 且 $\varepsilon=\mu^2$，$\mu\to 0$.

定理 2　若假设(H_1)－(H_3)成立，且存在定义在$(0,+\infty)$上单调不减的正值函数 $h(s)$，满足

$$\int_1^{+\infty}\frac{s\,ds}{h(s)}=+\infty$$

使得对$[a,c]\times\mathbb{R}^3$ 的紧子集中的(t,y,y',y'')及 $y'''\in\mathbb{R}$ 有

$$|f(t, y, y', y'', y''')| \leqslant h(|y'''|)$$

则两参数问题(1)—(5)对正的小参数 ε 和 μ，当 $\varepsilon=\mu^2(\mu\to 0)$ 时有解

$$y=y(t, \varepsilon_1)\in C^4[a, c]$$

满足

$$y^{(n)}(t, \varepsilon_1)=y_m^{(n)}(t, \varepsilon_1)+O(\varepsilon_1^{\ m+1}),\ n=0,1,2,\ t\in[a, c]$$

其中 $y_m(t, \varepsilon_1)$ 由(39)式给出，$\varepsilon_1=\mu$.

定理 2 证明过程与定理 1 类似，此处略去.

当 $\varepsilon=\mu^2(\mu\to 0)$ 时，问题(1)—(5)具有形如(39)式的渐近展开式，并在 $x=a$ 处和 $x=c$ 处附近各有一个薄层，其薄层宽度均为 $O(\varepsilon_1)$.

情形 3 当 $\frac{\mu^2}{\varepsilon}\to 0(\varepsilon\to 0)$ 时形式渐近解的构造.

令 $\varepsilon_1=\sqrt{\varepsilon}$，$\varepsilon_2=\frac{\mu}{\sqrt{\varepsilon}}$，则方程(1)转化为：

$$\varepsilon_1^2 y^{(4)}=f(t, y, y', y'', \varepsilon_1\varepsilon_2 y''') \tag{40}$$

方程(40)对应的外部解也可用与情形 1 相同的正则摄动方法得到：

$$\bar{Y}(t, \varepsilon_1, \varepsilon_2)\sim\sum_{i, j=0}^{\infty}\bar{Y}_{i,j}(t)\varepsilon_1^i\varepsilon_2^j$$

令

$$y(t, \varepsilon_1, \varepsilon_2)=\bar{Y}(t, \varepsilon_1, \varepsilon_2)+\varepsilon_1^3\bar{\mu}(\tau_5, \varepsilon_1, \varepsilon_2)$$

其中 $\tau_5=\frac{t-a}{\varepsilon_1}$ 为伸长变量，且

$$\bar{\mu}(\tau_5, \varepsilon_1, \varepsilon_2)\sim\sum_{i, j=0}^{\infty}\bar{\mu}_{i,j}(\tau_5)\varepsilon_1^i\varepsilon_2^j$$

其具有性质

$$\lim_{\tau_5\to+\infty}\bar{\mu}_{i,j}(\tau_5, \varepsilon_1, \varepsilon_2)=\lim_{\tau_5\to+\infty}\frac{\mathrm{d}\bar{\mu}_{i,j}}{\mathrm{d}\tau_5}(\tau_5, \varepsilon_1, \varepsilon_2)=\lim_{\tau_5\to+\infty}\frac{\mathrm{d}^2\bar{\mu}_{i,j}}{\mathrm{d}\tau_5^2}(\tau_5, \varepsilon_1, \varepsilon_2)=0 \tag{41}$$

则有

$$\frac{\mathrm{d}^4\bar{\mu}_{0,0}}{\mathrm{d}\tau_5^4}=f_{y''}(a, \bar{Y}_{0,0}(a), \bar{Y}'_{0,0}(a), \bar{Y}''_{0,0}(a), 0)\frac{\mathrm{d}^2\bar{\mu}_{0,0}}{\mathrm{d}\tau_5^2}=0 \tag{42}$$

$$\frac{\mathrm{d}^4\bar{\mu}_{i,j}}{\mathrm{d}\tau_5^4}=f_{y''}(a, \bar{Y}_{0,0}(a), \bar{Y}'_{0,0}(a), \bar{Y}''_{0,0}(a), 0)\frac{\mathrm{d}^2\bar{\mu}_{i,j}}{\mathrm{d}\tau_5^2}+\hat{F}_{i,j}(\tau_5),\ i+j\geqslant 1 \tag{43}$$

其中 $\hat{F}_{i,j}(\tau_5)(i+j\geqslant 1)$ 是 τ_5，$\bar{\mu}_{s,q}(s+q<i+j)$ 及其各阶导数的多项式函数.

令

$$y(t, \varepsilon_1, \varepsilon_2)=\bar{Y}(t, \varepsilon_1, \varepsilon_2)+\varepsilon_1^3\bar{\nu}(\tau_6, \varepsilon_1, \varepsilon_2)$$

其中 $\tau_6=\frac{c-t}{\varepsilon_1}$ 为伸长变量，且

$$\bar{\nu}(\tau_6, \varepsilon_1, \varepsilon_2)\sim\sum_{i, j=0}^{\infty}\bar{\nu}_{i,j}(\tau_6)\varepsilon_1^i\varepsilon_2^j$$

具有性质

$$\lim_{\tau_6\to+\infty}\bar{\nu}_{i,j}(\tau_6, \varepsilon_1, \varepsilon_2)=\lim_{\tau_6\to+\infty}\frac{\mathrm{d}\bar{\nu}_{i,j}}{\mathrm{d}\tau_6}(\tau_6, \varepsilon_1, \varepsilon_2)=\lim_{\tau_6\to+\infty}\frac{\mathrm{d}^2\bar{\nu}_{i,j}}{\mathrm{d}\tau_6^2}(\tau_6, \varepsilon_1, \varepsilon_2)=0 \tag{44}$$

则有

$$\frac{d^4\bar{\nu}_{0,0}}{d\tau_6^4}=f_{y''}(c,\bar{Y}_{0,0}(c),\bar{Y}'_{0,0}(c),\bar{Y}''_{0,0}(c),0)\frac{d^2\bar{\nu}_{0,0}}{d\tau_6^2} \tag{45}$$

$$\frac{d^4\bar{\nu}_{i,j}}{d\tau_6^4}=f_{y''}(c,\bar{Y}_{0,0}(c),\bar{Y}'_{0,0}(c),\bar{Y}''_{0,0}(c),0)\frac{d^2\bar{\nu}_{i,j}}{d\tau_6^2}+\check{F}_{i,j}(\tau_6),\ i+j\geqslant 1 \tag{46}$$

其中$\check{F}_{i,j}(\tau_6)$是关于τ_6，$\bar{\nu}_{s,q}(s+q<i+j)$及其各阶导数的多项式函数.

为了确定$\bar{\mu}_{i,j}(\tau_5)$，$\bar{\nu}_{i,j}(\tau_6)$所确定的定解条件，令

$$y(t,\varepsilon_1,\varepsilon_2)=\bar{Y}(t,\varepsilon_1,\varepsilon_2)+\varepsilon_1^3\bar{\mu}(\tau_5,\varepsilon_1,\varepsilon_2)+\varepsilon_1^3\bar{\nu}(\tau_6,\varepsilon_1,\varepsilon_2)$$

则有

$$g(\bar{Y}''_{0,0}(a),\bar{Y}''_{0,0}(b),\bar{Y}''_{0,0}(c),\bar{Y}'''_{0,0}(a)+\left.\frac{d^3\bar{\mu}_{0,0}}{d\tau_5^3}\right|_{\tau_5=0})=0 \tag{47}$$

$$h(\bar{Y}''_{0,0}(a),\bar{Y}''_{0,0}(b),\bar{Y}''_{0,0}(c),\bar{Y}'''_{0,0}(c)-\left.\frac{d^3\bar{\nu}_{0,0}}{d\tau_6^3}\right|_{\tau_6=0})=0 \tag{48}$$

$$g_\rho(\bar{Y}''_{0,0}(a),\bar{Y}''_{0,0}(b),\bar{Y}''_{0,0}(c),\bar{Y}'''_{0,0}(a)+\left.\frac{d^3\bar{\mu}_{0,0}}{d\tau_5^3}\right|_{\tau_5=0})\left.\frac{d^3\bar{\mu}_{i,j}}{d\tau_5^3}\right|_{\tau_5=0}=\bar{G}_{i,j}(\tau_5),\ i+j\geqslant 1 \tag{49}$$

$$h_\theta(\bar{Y}''_{0,0}(a),\bar{Y}''_{0,0}(b),\bar{Y}''_{0,0}(c),\bar{Y}'''_{0,0}(c)-\left.\frac{d^3\bar{\nu}_{0,0}}{d\tau_6^3}\right|_{\tau_6=0})\left.\frac{d^3\bar{\nu}_{i,j}}{d\tau_6^3}\right|_{\tau_6=0}=\bar{H}_{i,j}(\tau_6),\ i+j\geqslant 1 \tag{50}$$

其中$\bar{G}_{i,j}(\tau_5)$，$\bar{H}_{i,j}(\tau_6)$是依次确定的常数. 由假设(H_3)及(47)－(48)式可求出$\left.\frac{d^3\bar{\mu}_{0,0}}{d\tau_5^3}\right|_{\tau_5=0}$，记为$U_2$；$\left.\frac{d^3\bar{\nu}_{0,0}}{d\tau_6^3}\right|_{\tau_6=0}$，记为$V_2$. 再结合(41)－(46)，(49)，(50)式以及假设(H_2)可求出$\bar{\mu}_{i,j}(\tau_5)$，$\bar{\nu}_{i,j}(\tau_6)$，其中

$$\bar{\mu}_{0,0}(\tau_5)=\frac{U_2}{\lambda_5^3}e^{\lambda_5\tau_5}$$

$$\bar{\nu}_{0,0}(\tau_6)=\frac{V_2}{\lambda_6^3}e^{\lambda_6\tau_6}$$

λ_5为(42)式的特征方程的一个负根，λ_6为(45)式的特征方程的一个负根. 由$\bar{\mu}_{0,0}(\tau_5)$，$\bar{\nu}_{0,0}(\tau_6)$，$\hat{F}_{i,j}(\tau_5)$，$\check{F}_{i,j}(\tau_6)$的构造，可知$\bar{\mu}_{i,j}(\tau_5)$，$\bar{\nu}_{i,j}(\tau_6)$都具有指数型衰减的特征.

令

$$y_m(t,\varepsilon_1,\varepsilon_2)=\sum_{i,j=0}^m Y_{i,j}(t)\varepsilon_1^i\varepsilon_2^j+\psi(t-a)\varepsilon_1^3\sum_{i,j=0}^m\bar{\mu}_{i,j}\left(\frac{t-a}{\varepsilon_1}\right)\varepsilon_1^i\varepsilon_2^j+\psi(c-t)\varepsilon_1^3\sum_{i,j=0}^m\bar{\nu}_{i,j}\left(\frac{c-t}{\varepsilon_1}\right)\varepsilon_1^i\varepsilon_2^j \tag{51}$$

得到问题(1)－(5)的m阶形式渐近解，其中$\varepsilon_1=\sqrt{\varepsilon}$，$\varepsilon_2=\frac{\mu}{\sqrt{\varepsilon}}$且$\frac{\mu^2}{\varepsilon}\to 0$，$\varepsilon\to 0$.

定理 3　若假设(H_1)－(H_3)成立，且存在定义在$(0,+\infty)$上单调不减的正值函数$h(s)$，满足

$$\int_1^{+\infty}\frac{s\,ds}{h(s)}=+\infty$$

使得对$[a,c]\times\mathbb{R}^3$的紧子集中的(t,y,y',y'')及$y'''\in\mathbb{R}$有

$$|f(t,y,y',y'',y''')|\leqslant h(|y'''|)$$

则当$\frac{\mu^2}{\varepsilon}\to 0$，$\varepsilon\to 0$时，对正的小参数$\varepsilon$和$\mu$，两参数问题(1)～(5)有解

$$y=y(t,\varepsilon_1,\varepsilon_2)\in C^4[a,c]$$

满足

$$y^{(n)}(t,\varepsilon_1,\varepsilon_2)=y_m^{(n)}(t,\varepsilon_1,\varepsilon_2)+O(d_1^{m+1}),\ n=0,1,2,\ t\in[a,c]$$

其中$y_m(t,\varepsilon_1,\varepsilon_2)$由(51)式给出，$d_1=\max\{\varepsilon_1,\varepsilon_2\}$，$\varepsilon_1=\sqrt{\varepsilon}$，$\varepsilon_2=\frac{\mu}{\sqrt{\varepsilon}}$.

定理 3 证明过程与定理 1 类似，此处略去.

当$\frac{\mu^2}{\varepsilon} \to 0(\varepsilon \to 0)$时，问题(1)—(5)具有形如(51)式的渐近展开式，并在$x = a$处和$x = c$处附近各有一个薄层，其薄层宽度均为$O(\varepsilon_1)$.

3 应 用

考虑如下混合边值条件的双参数奇摄动问题

$$\varepsilon y^{(4)} = 6\mu y''' + 6y'' - y' - y - t,\ -1 < t < 1 \tag{52}$$

$$y(0) = 0 \tag{53}$$

$$y'(0) = 1 \tag{54}$$

$$4y''(-1) - y''(0) - y''(1) - y'''(-1) = 0 \tag{55}$$

$$-y''(-1) - y''(0) + 4y''(1) + y'''(1) = 0 \tag{56}$$

问题(52)—(56)满足假设[H_1]—[H_3]条件.

情形1′ 当参数ε, μ满足$\frac{\varepsilon}{\mu^2} \to 0$，$\mu \to 0$时的情形.

问题(52)—(56)的退化问题为

$$6Y''_{0,0} - Y'_{0,0} - Y_{0,0} - t = 0$$

$$Y_{0,0}(0) = 0$$

$$Y'_{0,0}(0) = 1$$

存在解$Y_{0,0}(t) = 2\mathrm{e}^{\frac{1}{2}t} - 3\mathrm{e}^{-\frac{1}{3}t} + 1 - t$.

在$x = -1$和$x = 1$处构造边界层校正项，有

$$6\frac{\mathrm{d}^3\hat{\mu}_{0,0}}{\mathrm{d}\tau_1^3} + 6\frac{\mathrm{d}^2\hat{\mu}_{0,0}}{\mathrm{d}\tau_1^2} = 0$$

$$\left.\frac{\mathrm{d}^3\hat{\mu}_{0,0}}{\mathrm{d}\tau_1^3}\right|_{\tau_1=0} = \frac{7}{4}\mathrm{e}^{-\frac{1}{2}} - \frac{13}{9}\mathrm{e}^{\frac{1}{3}} - \frac{1}{2}\mathrm{e}^{\frac{1}{2}} + \frac{1}{3}\mathrm{e}^{-\frac{1}{3}} - \frac{1}{6}$$

$$\frac{\mathrm{d}^4 v_{0,0}}{\mathrm{d}\tau_2^4} = -6\frac{\mathrm{d}^3 v_{0,0}}{\mathrm{d}\tau_2^3}$$

$$\left.\frac{\mathrm{d}^3 v_{0,0}}{\mathrm{d}\tau_2^3}\right|_{\tau_2=0} = -\frac{1}{2}\mathrm{e}^{-\frac{1}{2}} + \frac{9}{4}\mathrm{e}^{\frac{1}{2}} + \frac{1}{3}\mathrm{e}^{\frac{1}{3}} - \frac{11}{9}\mathrm{e}^{-\frac{1}{3}} - \frac{1}{6}$$

则

$$\hat{\mu}_{0,0}(\tau_1) = -U_0\mathrm{e}^{-\tau_1}$$

$$v_{0,0}(\tau_2) = -\frac{V_0}{216}\mathrm{e}^{-6\tau_2}$$

其中$U_0 = \left.\frac{\mathrm{d}^3\hat{\mu}_{0,0}}{\mathrm{d}\tau_1^3}\right|_{\tau_1=0}$，$V_0 = \left.\frac{\mathrm{d}^3 v_{0,0}}{\mathrm{d}\tau_2^3}\right|_{\tau_2=0}$.

因此问题(52)—(56)的零阶形式渐近解为

$$y_0(t, \varepsilon, \mu) = 2\mathrm{e}^{\frac{1}{2}t} - 3\mathrm{e}^{-\frac{1}{3}t} + 1 - t - \psi(t+1)\mu^3 U_0\mathrm{e}^{-\tau_1} - \psi(1-t)\frac{\varepsilon^3}{\mu^3}\frac{V_0}{216}\mathrm{e}^{-6\tau_2}$$

情形2′ 当参数ε, μ满足$\varepsilon = \mu^2, \mu \to 0$时的情形.

类似情形1′，问题(52)—(56)的退化问题存在解$Y_0(t) = 2\mathrm{e}^{\frac{1}{2}t} - 3\mathrm{e}^{-\frac{1}{3}t} + 1 - t$.

在$x = -1$和$x = 1$处构造边界层校正项，有

$$\frac{\mathrm{d}^4\tilde{\mu}_0}{\mathrm{d}\tau_3^4} = 6\frac{\mathrm{d}^3\tilde{\mu}_0}{\mathrm{d}\tau_3^3} + 6\frac{\mathrm{d}^2\tilde{\mu}_0}{\mathrm{d}\tau_3^2}$$

$$\left.\frac{d^3\tilde{\mu}_0}{d\tau_3^3}\right|_{\tau_3=0}=\frac{7}{4}e^{-\frac{1}{2}}-\frac{13}{9}e^{\frac{1}{3}}-\frac{1}{2}e^{\frac{1}{2}}+\frac{1}{3}e^{-\frac{1}{3}}-\frac{1}{6}$$

$$\frac{d^4\tilde{\nu}_0}{d\tau_4^4}=-6\frac{d^3\tilde{\nu}_0}{d\tau_4^3}+6\frac{d^2\tilde{\nu}_0}{d\tau_4^2}$$

$$\left.\frac{d^3\tilde{\nu}_0}{d\tau_4{}^3}\right|_{\tau_4=0}=-\frac{1}{2}e^{-\frac{1}{2}}+\frac{9}{4}e^{\frac{1}{2}}+\frac{1}{3}e^{\frac{1}{3}}-\frac{11}{9}e^{-\frac{1}{3}}-\frac{1}{6}$$

则

$$\tilde{\mu}_0(\tau_3)=\frac{U_1}{(3-\sqrt{15})^3}e^{(3-\sqrt{15})\tau_3}$$

$$\tilde{\nu}_0(\tau_4)=\frac{V_1}{(-3-\sqrt{15})^3}e^{(-3-\sqrt{15})\tau_4}$$

其中 $U_1=\left.\frac{d^3\tilde{\mu}_0}{d\tau_3^3}\right|_{\tau_3=0}$，$V_1=\left.\frac{d^3\tilde{\nu}_0}{d\tau_4^3}\right|_{\tau_4=0}$.

因此问题(52)－(56)的零阶形式渐近解为

$$y_0(t,\mu)=2e^{\frac{1}{2}t}-3e^{-\frac{1}{3}t}+1-t+\psi(t+1)\mu^3\frac{U_1}{(3-\sqrt{15})^3}e^{(3-\sqrt{15})\tau_3}+\psi(1-t)\mu^3\frac{V_1}{(-3-\sqrt{15})^3}e^{(-3-\sqrt{15})\tau_4}$$

情形 3′　当参数 ε,μ 满足 $\frac{\mu^2}{\varepsilon}\to 0,\varepsilon\to 0$ 时的情形.

类似情形 1′，问题(52)－(56)的退化问题存在解 $\bar{Y}_{0,0}(t)=2e^{\frac{1}{2}t}-3e^{-\frac{1}{3}t}+1-t$.

在 $x=-1$ 和 $x=1$ 处构造边界层校正项，有

$$\frac{d^4\bar{\mu}_{0,0}}{d\tau_5^4}=6\frac{d^2\bar{\mu}_{0,0}}{d\tau_5^2}$$

$$\left.\frac{d^3\bar{\mu}_{0,0}}{d\tau_5^3}\right|_{\tau_5=0}=\frac{7}{4}e^{-\frac{1}{2}}-\frac{13}{9}e^{\frac{1}{3}}-\frac{1}{2}e^{\frac{1}{2}}+\frac{1}{3}e^{-\frac{1}{3}}-\frac{1}{6}$$

$$\frac{d^4\bar{\nu}_{0,0}}{d\tau_6^4}=6\frac{d^2\bar{\nu}_{0,0}}{d\tau_6^2}$$

$$\left.\frac{d^3\bar{\nu}_{0,0}}{d\tau_6^3}\right|_{\tau_6=0}=-\frac{1}{2}e^{-\frac{1}{2}}+\frac{9}{4}e^{\frac{1}{2}}+\frac{1}{3}e^{\frac{1}{3}}-\frac{11}{9}e^{-\frac{1}{3}}-\frac{1}{6}$$

则

$$\bar{\mu}_{0,0}(\tau_5)=-\frac{U_2}{6\sqrt{6}}e^{-\sqrt{6}\tau_5}$$

$$\bar{\nu}_{0,0}(\tau_6)=-\frac{V_2}{6\sqrt{6}}e^{-\sqrt{6}\tau_6}$$

其中 $U_2=\left.\frac{d^3\bar{\mu}_{0,0}}{d\tau_5^3}\right|_{\tau_5=0}$，$V_2=\left.\frac{d^3\bar{v}_{0,0}}{d\tau_6^3}\right|_{\tau_6=0}$.

因此问题(52)－(56)的零阶形式渐近解为

$$y_0(t,\varepsilon,\mu)=2e^{\frac{1}{2}t}-3e^{-\frac{1}{3}t}+1-t-\psi(t+1)(\sqrt{\varepsilon})^3\frac{U_2}{6\sqrt{6}}e^{-\sqrt{6}\tau_5}-\psi(1-t)(\sqrt{\varepsilon})^3\frac{V_2}{6\sqrt{6}}e^{-\sqrt{6}\tau_6}$$

参考文献：

［1］ FENG Y H，CHEN H J，MO J Q. A Class of Singular Perturbation Problem for the Nonlinear Differential-Integral Disturbed Evolution Equation［J］. Wuhan University Journal of Natural Sciences，2018，23(4)：318-322.

［2］ WANGJ，LIANGK，HUANGX，et al. Dissipative Fault-Tolerant Control for Nonlinear Singular Perturbed Systems with Markov Jumping Parameters Based on Slow State Feedback [J]. Applied Mathematics and Computation, 2018, 328: 247-262.

[3] XIE F. On a Class of Singular Boundary Value Problems with Singular Perturbation [J]. Journal of Differential Equations, 2012, 252(3): 2370-2387.

[4] O'MALLEY. Introduction to Singular Perturbations [M]. New York: Academic Press, 1974.

[5] 莫嘉琪. 一类两参数半线性奇摄动问题解的渐近性态 [J]. 应用数学学报, 2009, 32(5): 903-911.

[6] 欧阳成. 一类四阶方程两参数的奇摄动问题 [J]. 应用数学学报, 2009, 32(4): 640-647.

[7] 吴有萍, 姚静荪, 庄红艳. 一类具非线性边值条件的双参数奇摄动问题 [J]. 高校应用数学学报 A 辑, 2011, 26(3): 288-294.

[8] 周克浩, 杨雪洁, 姚静荪. 一类双参数奇摄动边值问题的内层解 [J]. 吉林大学学报(理学版), 2014, 52(2): 201-204.

[9] 史娟荣, 汪维刚, 吴钦宽, 等. 两参数非线性奇摄动燃烧问题的渐近解 [J]. 兰州大学学报(自然科学版), 2016, 52(1): 107-111.

[10] 占小丽, 余赞平, 周哲彦. 两类带非线性混合边界条件的四阶微分方程三点边值问题 [J]. 福建师范大学学报(自然科学版), 2013, 29(1): 15-19.

A Class of Two-Parameter Singularly Perturbed Problems with Mixed Boundary Value Conditions

LIU Yan

Department of Electronic Engineering, Wanjiang College of Anhui Normal University, Wuhu Anhui 241000, China

Abstract: A class of singularly perturbed problems with two parameters for fourth-order differential equations with nonlinear mixed boundary value conditions have been discussed. Under appropriate conditions, the formal asymptotic solutions have been constructed under three different cases which the two small parameters are correlative using the composite expansion method. According to the theory of differential inequalities, the existence of solutions and the uniform validity of the asymptotic solutions in the three cases are proved.

Key words: singular perturbation; nonlinear mixed boundary value conditions; two parameters; the theory of differential inequalities

责任编辑 张 枸