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带非局部边界条件的反应扩散方程组的爆破现象①

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摘要: 研究了一类带非局部边界条件的非线性反应扩散方程组解的爆破问题. 通过构造恰当的辅助函数,结合改进的微分不等式技巧,建立了解在有限时间爆破的充分条件,得到了爆破时间 t^* 的上界估计;若爆破发生,相应可得 t^* 的下界估计.

关 键 词: 反应扩散方程;非局部边界条件;爆破;上下界

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On Blow-up Phenomena in a Reaction-Diffusion Equation Systems with Nonlocal Boundary Conditions

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Abstract: In this paper, the blow-up of solutions has been investigated to a class of nonlinear reaction-diffusion systems with non-local boundary conditions. By constructing appropriate auxiliary functions and improved differential inequality techniques, sufficient conditions have been established to blow up at some finite time, and then an upper bound of the blow-up time t^* has been obtained; moreover, if the blow-up occurred, the lower bound estimation of the blow-up time t^* will be got.

Key words: reaction-diffusion systems; non-local boundary conditions; blow-up; upper and lower bounds

本文主要考虑如下带有非局部边界条件及非局部源的非线性反应扩散方程组的初边值问题

$$
\begin{cases}
u_t = \nabla \cdot (\rho_1(u)\nabla u) + k_1(t) f_1(v),\ v_t = \nabla \cdot (\rho_2(v)\nabla v) + k_2(t) f_2(u) & x \in D \times (0,\ t^*) \\
\dfrac{\partial u}{\partial \upsilon} = k_3(t)\displaystyle\int_D g_1(u)\mathrm{d}x,\ \dfrac{\partial v}{\partial \upsilon} = k_4(t)\displaystyle\int_D g_2(v)\mathrm{d}x & x \in \partial D \times (0,\ t^*) \\
u(x,\ 0) = u_0(x) \geqslant 0,\ v(x,\ 0) = v_0(x) \geqslant 0 & x \in \overline{D}
\end{cases}
\tag{1}
$$

其中:$D \subset \mathbb{R}^n (n \geqslant 2)$ 是非空带有光滑边界 ∂D 的有界凸域,υ 是相对于 ∂D 的向外法向量,t^* 为爆破发生

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时的爆破时间；f_1, f_2, g_1, g_2 是属于 $C^1(\overline{\mathbb{R}_+})$ 的非负函数；k_1, k_2, k_3, k_4 是属于 $C^1(\overline{\mathbb{R}_+})$ 的有界正函数；ρ_1, ρ_2 是属于 $C^1(\overline{\mathbb{R}_+})$ 的正函数，$u_0(x), v_0(x), x \in \overline{D}$ 是 $C^1(\overline{D})$ 上不恒为 0 的非负函数且满足兼容性条件. 由文献[13]可知(1)式的非负古典解.

近年来，关于非线性抛物方程(方程组)的全局解的存在与非存在性、解的爆破情况、爆破时间上下界、解的渐近行为等的研究不断涌现[7-12]. 带非局部边界条件的反应扩散问题基于众多物理意义，如热弹性理论，在该理论中，解可描述为每一体积物质的熵. 所以，目前研究者们开始了对这类问题的研究[1-4]. 但目前还没有对于带非局部边界条件及非局部源的反应扩散方程组的爆破研究，故本文将致力于研究此类方程组的爆破现象.

受上述工作的启发，我们将在本文中研究方程(1)的爆破现象. 结合恰当的微分不等式技巧，建立了在有限时间内解爆破的适当条件，还得到了爆破时间的上界和下界估计.

定理 1 若(u, v) 是方程(1)的非负古典解，$D \subset \mathbb{R}^n (n \geqslant 2)$，设函数 $f_1, f_2, g_1, g_2, \rho_1, \rho_2, k_1, k_2$ 满足

$$f_1(s) \geqslant s^{p_1},\ f_2(s) \geqslant s^{p_2},\ g_1(s) \geqslant s^{q_1},\ g_2(s) \geqslant s^{q_2},\ \rho_1(s) \geqslant b_1,\ \rho_2(s) \geqslant b_2,\ s \geqslant 0$$

$$\frac{k_1'(t)}{k_1(t)} \geqslant \xi_1,\ \frac{k_2'(t)}{k_2(t)} \geqslant \xi_2,\ t \geqslant 0 \tag{2}$$

其中：ξ_1, ξ_2 为非负常数；$p_1, p_2, q_1, q_2, b_1, b_2$ 为正常数且满足 $q_1 > p_2 > 1$，$q_2 > p_1 > 1$. 现假设 $p_2 \geqslant p_1$ 且初始条件 u_0 满足

$$C_1 B(0) + 2^{1-p_1} M B^{p_1}(0) - M \frac{p_2 - p_1}{p_2}\left(\frac{p_1}{p_2}\right)^{\frac{p_1}{p_2-p_1}} > 0 \tag{3}$$

其中

$$C_1 = \min\left\{\frac{\xi_1}{p_1 - 1},\ \frac{\xi_2}{p_2 - 1}\right\}$$

$$M = \inf_{t \geqslant 0}\{\max(m_1(t),\ m_2(t))\} > 0$$

$$m_1(t) = b_1 \mid \partial D \mid \mid D \mid^{1-q_1} k_3(t) k_1^{\frac{1-q_1}{p_1-1}}(t) \Phi^{q_1-p_2}(0) + k_2^{\frac{p_2}{p_2-1}}(t) k_1^{\frac{p_2}{1-p_1}}(t) \mid D \mid^{1-p_2}$$

$$m_2(t) = b_2 \mid \partial D \mid \mid D \mid^{1-q_2} k_4(t) k_2^{\frac{1-q_2}{p_2-1}}(t) \Psi^{q_2-p_1}(0) + k_1^{\frac{p_1}{p_1-1}}(t) k_2^{\frac{p_1}{1-p_2}}(t) \mid D \mid^{1-p_1}$$

则在某一 t^* 时刻，解(u, v) 在 $B(t)$ 的测度下爆破.

证 现构造辅助函数如下

$$B(t) = \Phi(t) + \Psi(t) \qquad \Phi(t) = k_1^{\frac{1}{p_1-1}}(t)\int_D u\,\mathrm{d}x \qquad \Psi(t) = k_2^{\frac{1}{p_2-1}}(t)\int_D v\,\mathrm{d}x \qquad t \geqslant 0 \tag{4}$$

由散度定理及条件(2)得

$$\begin{aligned} B'(t) \geqslant & C_1 B(t) + b_1 \mid \partial D \mid k_1^{\frac{1}{p_1-1}}(t) k_3(t)\int_D u^{q_1}\,\mathrm{d}x + b_2 \mid \partial D \mid k_2^{\frac{1}{p_2-1}}(t) k_4(t)\int_D v^{q_2}\,\mathrm{d}x + \\ & k_1^{\frac{p_1}{p_1-1}}(t)\int_D v^{p_1}\,\mathrm{d}x + k_2^{\frac{p_2}{p_2-1}}(t)\int_D u^{p_2}\,\mathrm{d}x \end{aligned} \tag{5}$$

考虑对 $\Phi(t)$，$\Psi(t)$ 应用 Hölder 不等式，可进一步得

$$k_1^{\frac{1}{p_1-1}}(t)\int_D u^{q_1}\,\mathrm{d}x \geqslant \mid D \mid^{1-q_1} k_1^{\frac{1-q_1}{p_1-1}}(t) \Phi^{q_1}(t) \qquad k_2^{\frac{1}{p_2-1}}(t)\int_D v^{q_2}\,\mathrm{d}x \geqslant \mid D \mid^{1-q_2} k_2^{\frac{1-q_2}{p_2-1}}(t) \Psi^{q_2}(t) \tag{6}$$

再次应用 Hölder 不等式得

$$\int_D u^{p_2}\,\mathrm{d}x \geqslant \mid D \mid^{1-p_2} k_1^{\frac{p_2}{1-p_1}}(t) \Phi^{p_2}(t) \qquad \int_D v^{p_1}\,\mathrm{d}x \geqslant \mid D \mid^{1-p_1} k_2^{\frac{p_1}{1-p_2}}(t) \Psi^{p_1}(t) \tag{7}$$

将(6),(7)式带入到(5)式中，得

$$B'(t) \geqslant C_1 B(t) + b_1 \mid \partial D \mid \mid D \mid^{1-q_1} k_1^{\frac{1-q_1}{p_1-1}}(t) k_3(t) \Phi^{q_1}(t) + b_2 \mid \partial D \mid \mid D \mid^{1-q_2} k_2^{\frac{1-q_2}{p_2-1}}(t) k_4(t) \Psi^{q_2}(t) +$$

$$|D|^{1-p_2}k_1^{\frac{p_2}{1-p_1}}(t)k_2^{\frac{p_2}{p_2-1}}(t)\Phi^{p_2}(t)+|D|^{1-p_1}k_2^{\frac{p_1}{1-p_2}}(t)k_1^{\frac{p_1}{p_1-1}}(t)\Psi^{p_1}(t) \tag{8}$$

这表明对所有 $t\in(0,t^*)$，$B(t)$，$\Phi(t)$，$\Psi(t)$ 均为非减函数. 由(4)式知 $\Phi(t)\geqslant\Phi(0)>0$，$\Psi(t)\geqslant\Psi(0)>0$. 由 $q_1>p_2>1$，$q_2>p_1>1$ 可推出 $\Phi^{q_1}(t)\geqslant\Phi^{p_2}(t)\Phi^{q_1-p_2}(0)$，$\Psi^{q_2}(t)\geqslant\Psi^{p_1}(t)\Psi^{q_2-p_1}(0)$，再代入(8)式中，结合 M 的定义知

$$B'(t)\geqslant C_1B(t)+M(\Phi^{p_2}(t)+\Psi^{p_1}(t)) \tag{9}$$

现分两步完成证明.

第一步，当 $p_1=p_2$ 时，由不等式 $d_1^a+d_2^a\geqslant 2^{1-a}(d_1+d_2)^a$，$d_1$，$d_2>0$，$a>1$ 得 $B'(t)\geqslant C_1B(t)+2^{1-p_1}MB^{p_1}(t)$，再结合常微分方程的知识对其由 0 到 t 积分，即有

$$(B(t))^{1-p_1}\leqslant \mathrm{e}^{(1-p_1)C_1t}\left(B^{1-p_1}(0)+\frac{2^{1-p_1}M}{C_1}\right)-\frac{2^{1-p_1}M}{C_1}=\phi(t) \tag{10}$$

结合(3)式，容易发现 $\phi(T)=0$ 且 $\phi(T)<0$，$t>T$ ，进而得到

$$t^*\leqslant T=\frac{1}{(p_1-1)C_1}\ln\left(1+\frac{B^{p_1-1}(0)C_1}{2^{p_1-1}M}\right) \tag{11}$$

第二步，当 $p_2>p_1$，运用 Young 不等式可得

$$\Phi^{p_1}(t)\leqslant\Phi^{p_2}(t)+\frac{p_2-p_1}{p_2}\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1-p_2}}\qquad \Phi^{p_2}(t)\geqslant\Phi^{p_1}(t)-\frac{p_2-p_1}{p_2}\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1-p_2}} \tag{12}$$

将(12)式代入(9)式中，结合 $d_1^a+d_2^a\geqslant 2^{1-a}(d_1+d_2)^a$，$d_1,d_2>0$，$a>1$ 得

$$B'(t)\geqslant C_1B(t)+2^{1-p_1}MB^{p_1}(t)-M\frac{p_2-p_1}{p_2}\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1-p_2}}=\psi(B(t)) \tag{13}$$

注意到(3)式表明 $\psi(B(t))>0$，$t\geqslant0$，易知

$$\psi'(B(t))=C_1+2^{1-p_1}Mp_1B^{p_1-1}(t) \tag{14}$$

因此易得当 $t\geqslant0$ ，有 $\psi'(B(t))>0$ 成立，即 $\psi(B(t))$ 是关于 $B(t)$ 递增的函数. 故由(3)－(13)式可得 $\psi(B(t))>\psi(B(0))>0$. 又由(13)式得解 (u,v) 在 t^* 时，在 $B(t)$ 的测度下爆破. 若 $\lim_{t\to t^*}B(t)=\infty$，对式(13)式两边关于时间 t 在$[0,t^*]$上积分，则有

$$t^*\leqslant\int_{B(0)}^{+\infty}\frac{\mathrm{d}\eta}{C_1\eta+2^{1-p_1}M\eta^{p_1}-M\frac{p_2-p_1}{p_2}\left(\frac{p_1}{p_2}\right)^{\frac{p_1}{p_2-p_1}}} \tag{15}$$

至此，定理 1 证明完毕.

定理 2　若 (u,v) 是方程(1)的非负古典解，为得到 t^* 下界，现限制 $D\subset\mathbb{R}^n(n\geqslant3)$，并假设函数 f_1，f_2，g_1，g_2，ρ_1，ρ_2 和 k_1，k_2 满足

$$\begin{gathered}0\leqslant f_1(s)\leqslant s^{p_1},\ 0\leqslant f_2(s)\leqslant s^{p_2},\ 0\leqslant g_1(s)\leqslant s^{q_1},\ 0\leqslant g_2(s)\leqslant s^{q_2},\\ b_1\leqslant\rho_1(s)\leqslant b_2+b_3s^{l_1},\ c_1\leqslant\rho_2(s)\leqslant c_2+c_3s^{l_2},\ \frac{k_1'(t)}{k_1(t)}\leqslant\alpha_1,\ \frac{k_2'(t)}{k_2(t)}\leqslant\alpha_2\end{gathered} \tag{16}$$

其中：s,t 为非负常数；α_1,α_2 是非负函数；$p_1,p_2,q_1,q_2,b_1,b_2,b_3,c_1,c_2,c_3,l_1,l_2$ 为正常数，满足 $q_1,q_2>1$，$l_1,l_2>1$ 且

$$p_1>\max\left\{1+\frac{2}{n},\ l_1+\frac{l_1+q_1-1}{n}\right\}\qquad p_2>\max\left\{1+\frac{2}{n},\ l_2+\frac{l_2+q_2-1}{n}\right\} \tag{17}$$

若 u 在某 t^* 时刻，在测度 $G(t)$ 下无界，则 t^* 的下界为 $t^*\geqslant\int_{G(0)}^{+\infty}\frac{\mathrm{d}\tau}{M_1\tau+M_2\tau^3+M_3}$，其中

$$M_1=\sup_{t\geqslant0}\{\max(A_1(t),A_2(t))\},\ M_2=\sup_{t\geqslant0}\{\max(B_1(t),B_2(t))\}$$

$$M_3=\frac{n(p-p_1)}{n+1}|D|k_1^{n+1}(t)+\frac{n(p-p_2)}{n+1}|D|k_2^{n+1}(t)$$

证 首先构造辅助函数如下：

$$G(t)=\Phi(t)+\Psi(t)=k_1^n(t)\int_D u^{n(p-1)}\mathrm{d}x+k_2^n(t)\int_D v^{n(p-1)}\mathrm{d}x,\ t\geqslant 0 \tag{18}$$

其中 $p=\max\{p_1,p_2\}$，由文献[6]可得 Sobolev 不等式

$$\left(\int_D (u^{\frac{n(p-1)}{2}})^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{2n}}\leqslant C\left(\int_D u^{n(p-1)}\mathrm{d}x+\int_D |\nabla u^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\right)^{\frac{1}{2}} \tag{19}$$

其中 $C=C(n,D)$ 是依赖 n 和 D 的 Sobolev 嵌入常量. 对 $G(t)$ 求导，由散度定理和条件(16)得到

$$\begin{aligned}G'(t)\leqslant & n\alpha_1\Phi(t)+b_2n(p-1)k_1^n(t)k_3(t)\int_{\partial D}u^{n(p-1)-1}\mathrm{d}s\int_D u^{q_1}\mathrm{d}x+n(p-1)k_1^{n+1}(t)\int_D u^{n(p-1)-1}v^{p_1}\mathrm{d}x+\\ & b_3n(p-1)k_1^n(t)k_3(t)\int_{\partial D}u^{n(p-1)+l_1-1}\mathrm{d}s\int_D u^{q_1}\mathrm{d}x-\frac{4b_1[n(p-1)-1]}{n(p-1)}k_1^n(t)\int_D|\nabla u^{\frac{n(p-1)}{2}}|^2\mathrm{d}x+\\ & n\alpha_2\Psi(t)+c_2n(p-1)k_2^n(t)k_4(t)\int_{\partial D}v^{n(p-1)-1}\mathrm{d}s\int_D v^{q_2}\mathrm{d}x+n(p-1)k_2^{n+1}(t)\int_D v^{n(p-1)-1}u^{p_2}\mathrm{d}x+\\ & c_3n(p-1)k_2^n(t)k_4(t)\int_{\partial D}v^{n(p-1)+l_2-1}\mathrm{d}s\int_D v^{q_2}\mathrm{d}x-\frac{4c_1[n(p-1)-1]}{n(p-1)}k_2^n(t)\int_D|\nabla v^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\end{aligned} \tag{20}$$

接下来分四步完成证明.

第一步，估计(20)式右端的第二和第七项，对第二项使用散度定理，有下列不等式[5]

$$k_1^n(t)\int_{\partial D}u^{n(p-1)-1}\mathrm{d}s\leqslant\frac{n}{\rho_0}k_1^n(t)\int_D u^{n(p-1)-1}\mathrm{d}x+\frac{[n(p-1)-1]d}{\rho_0}k_1^n(t)\int_D u^{n(p-1)-2}|\nabla u|\mathrm{d}x \tag{21}$$

其中：$\rho_0=\min_{x\in\partial D}(x\cdot\nu)$，$d=\max_{x\in\overline{D}}|x|$. 现对(21)右端两项分别应用 Hölder 不等式，再回代其中. 由(16)，(17)式可得 $0<\dfrac{q_1}{n(p_1-1)}<1$，再次运用 Hölder 不等式，得到

$$\int_D u^{q_1}\mathrm{d}x\leqslant|D|^{1-\frac{q_1}{n(p-1)}}\left(\int_D u^{n(p-1)}\mathrm{d}x\right)^{\frac{q_1}{n(p-1)}}=|D|^{1-\frac{q_1}{n(p-1)}}k_1^{\frac{q_1}{1-p}}(t)\Phi^{\frac{q_1}{n(p-1)}}(t) \tag{22}$$

由(21)，(22)式以及 Young 不等式，可分别得到

$$\begin{aligned}& k_1^n(t)\int_{\partial D}u^{n(p-1)-1}\mathrm{d}s\int_D u^{q_1}\mathrm{d}x\leqslant\\ & |D|^{1+\frac{1-q_1}{n(p-1)}}k_1^{\frac{1-q_1}{p-1}}(t)\left[\frac{n}{\rho_0}\Phi^{1+\frac{q_1-1}{n(p-1)}}(t)+\frac{[n(p-1)-1]d}{n(p-1)\rho_0}\left(\frac{1}{\varepsilon_1}\Phi^{1+\frac{2(q_1-1)}{n(p-1)}}(t)+\varepsilon_1k_1^n(t)\int_D|\nabla u^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\right)\right]\end{aligned} \tag{23}$$

$$\begin{aligned}& k_2^n(t)\int_{\partial D}v^{n(p-1)-1}\mathrm{d}s\int_D v^{q_2}\mathrm{d}x\leqslant\\ & |D|^{1+\frac{1-q_2}{n(p-1)}}k_2^{\frac{1-q_2}{p-1}}(t)\left[\frac{n}{\rho_0}\Psi^{1+\frac{q_2-1}{n(p-1)}}(t)+\frac{[n(p-1)-1]d}{n(p-1)\rho_0}\left(\frac{1}{\theta_1}\Psi^{1+\frac{2(q_2-1)}{n(p-1)}}(t)+\theta_1k_2^n(t)\int_D|\nabla v^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\right)\right]\end{aligned} \tag{24}$$

其中 $\varepsilon_1=\dfrac{b_1\rho_0}{b_2nd(p-1)|D|^{1+\frac{1-q_1}{n(p-1)}}k_1^{\frac{1-q_1}{p-1}}(t)k_3(t)}$，$\theta_1=\dfrac{c_1\rho_0}{c_2nd(p-1)|D|^{1+\frac{1-q_2}{n(p-1)}}k_2^{\frac{1-q_2}{p-1}}(t)k_4(t)}$.

第二步，考虑(20)式右侧第四和第九项. 由散度定理，Hölder 不等式及 Young 不等式，得

$$\begin{aligned}& \Phi^{\frac{q_1}{n(p-1)}}(t)k_1^n(t)\int_{\partial D}u^{n(p-1)+l_1-1}\mathrm{d}s\leqslant\\ & \frac{1}{2}\Phi(t)+\frac{1}{2\rho_0^2}\left(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\right)\Phi^{\frac{2q_1}{n(p-1)}}(t)k_1^n(t)\int_D u^{n(p-1)+2(l_1-1)}\mathrm{d}x+\\ & \frac{2\varepsilon_2}{n^2(p-1)^2}k_1^n(t)\int_D|\nabla u^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\end{aligned}$$

又由(16)，(17)式得 $0<\dfrac{(n-2)(l_1-1)}{n(p-1)}<1$，$0<\dfrac{(n-2)(l_2-1)}{n(p-1)}<1$ 和 $0<\dfrac{(l_1-1)}{p-1}<1$，$0<\dfrac{(l_2-1)}{p-1}<1$，结合 Hölder 不等式，Sobolev 不等式(19)以及不等式 $(a+b)^\mu\leqslant a^\mu+b^\mu$，$a\geqslant 0$，$b\geqslant 0$，$0<\mu\leqslant 1$，进

而将(25)式右侧第二项化为与 $\Phi(t)$ 相关的式子，再结合(22)式推导出

$$k_1^n(t)\int_{\partial D}u^{n(p-1)+l_1-1}\mathrm{d}s\int_D u^{q_1}\mathrm{d}x\leqslant$$

$$|D|^{1-\frac{q_1}{n(p-1)}}k_1^{\frac{q_1}{1-p}}(t)\Big[\frac{1}{2}\Phi(t)+\frac{1}{2\rho_0^2}\Big(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\Big)C^{\frac{2(l_1-1)}{p-1}}k_1^{\frac{2(1-l_1)}{p-1}}(t)\Phi^{1+\frac{2(l_1+q_1-1)}{n(p-1)}}(t)+$$

$$\frac{p-l_1}{2{\rho_0}^2(p-1)}\Big(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\Big)\varepsilon_3^{\frac{1-l_1}{p-l_1}}C^{\frac{2(l_1-1)}{p-1}}k_1^{\frac{2(1-l_1)}{p-1}}(t)\Phi^{1+\frac{2(l_1+q_1-1)}{n(p-l_1)}}(t)+$$

$$\Big(\frac{2\varepsilon_2}{n^2(p-1)^2}+\frac{(l_1-1)\varepsilon_3}{2\rho_0^2(p-1)}\Big(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\Big)C^{\frac{2(l_1-1)}{p-1}}k_1^{\frac{2(1-l_1)}{p-1}}(t)\Big)\times k_1^n(t)\int_D|\nabla u^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\Big] \tag{26}$$

类似可得

$$k_2^n(t)\int_{\partial D}v^{n(p-1)+l_2-1}\mathrm{d}s\int_D v^{q_2}\mathrm{d}x\leqslant$$

$$|D|^{1-\frac{q_2}{n(p-1)}}k_2^{\frac{q_2}{1-p}}(t)\Big[\frac{1}{2}\Psi(t)+\frac{1}{2\rho_0^2}\Big(n^2+\frac{[n(p-1)+l_2-1]^2d^2}{\theta_2}\Big)C^{\frac{2(l_2-1)}{p-1}}k_2^{\frac{2(1-l_2)}{p-1}}(t)\Psi^{1+\frac{2(l_2+q_2-1)}{n(p-1)}}(t)+$$

$$\frac{p-l_2}{2\rho_0^2(p-1)}\Big(n^2+\frac{[n(p-1)+l_2-1]^2d^2}{\theta_2}\Big)\theta_3^{\frac{1-l_2}{p-l_2}}C^{\frac{2(l_2-1)}{p-1}}k_2^{\frac{2(1-l_2)}{p-1}}(t)\Psi^{1+\frac{2(l_2+q_2-1)}{n(p-l_2)}}(t)+$$

$$\Big(\frac{2\theta_2}{n^2(p-1)^2}+\frac{(l_2-1)\theta_3}{2\rho_0^2(p-1)}\Big(n^2+\frac{[n(p-1)+l_2-1]^2d^2}{\theta_2}\Big)C^{\frac{2(l_2-1)}{p-1}}k_2^{\frac{2(1-l_2)}{p-1}}(t)\Big)\times k_2^n(t)\int_D|\nabla v^{\frac{n(p-1)}{2}}|^2\mathrm{d}x\Big] \tag{27}$$

其中

$$\varepsilon_2=\frac{b_1[n(p-1)-1]}{2b_3|D|^{1-\frac{q_1}{n(p-1)}}k_1^{\frac{q_1}{1-p}}(t)k_3(t)}$$

$$\varepsilon_3=\frac{2b_1\rho_0^2[n(p-1)-1]C^{\frac{2(1-l_1)}{p-1}}|D|^{\frac{q_1}{n(p-1)}-1}}{b_3n^2(p-1)(l_1-1)\Big(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\Big)k_1^{\frac{2(l_1-1)-q_1}{p-1}}(t)k_3(t)}$$

$$\theta_2=\frac{c_1[n(p-1)-1]}{2c_3|D|^{1-\frac{q_2}{n(p-1)}}k_2^{\frac{q_2}{1-p}}(t)k_4(t)}$$

$$\theta_3=\frac{2c_1\rho_0^2[n(p-1)-1]C^{\frac{2(1-l_2)}{p-1}}|D|^{\frac{q_2}{n(p-1)}-1}}{c_3n^2(p-1)(l_2-1)\Big(n^2+\frac{[n(p-1)+l_2-1]^2d^2}{\theta_2}\Big)k_2^{\frac{2(l_2-1)-q_2}{p-1}}(t)k_4(t)}$$

第三步，估计(20)式右侧第三项和第八项．首先，对第三项运用 Hölder 不等式和 Young 不等式得到

$$\int_D u^{n(p-1)-1}v^{p_1}\mathrm{d}x\leqslant\frac{n(p-1)-1}{n(p-1)+p_1-1}\int_D u^{n(p-1)+p_1-1}\mathrm{d}x+\frac{p_1}{n(p-1)+p_1-1}\int_D v^{n(p-1)+p_1-1}\mathrm{d}x \tag{28}$$

再对(28)式右端两项分别使用 Hölder 不等式，而后回代入(28)式中．对于第八项，采取类似的估计方法，再联立两项所得估计结果推出

$$\begin{aligned}&n(p-1)k_1^{n+1}(t)\int_D u^{n(p-1)-1}v^{p_1}\mathrm{d}x+n(p-1)k_2^{n+1}(t)\int_D v^{n(p-1)-1}u^{p_2}\mathrm{d}x\leqslant\\&\frac{n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2}{(n+1)(p-1)}\Big(n(p-1)k_1^{n+1}(t)\int_D u^{(n+1)(p-1)}\mathrm{d}x\Big)+\\&\frac{n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1}{(n+1)(p-1)}\Big(n(p-1)k_2^{n+1}(t)\int_D v^{(n+1)(p-1)}\mathrm{d}x\Big)+\\&\frac{n(p-p_1)}{n+1}|D|k_1^{n+1}(t)+\frac{n(p-p_2)}{n+1}|D|k_2^{n+1}(t)\end{aligned} \tag{29}$$

现对(29)式右端第一、二项运用 Hölder 不等式，Sobolev 不等式(19)及 Young 不等式，进而得

$$n(p-1)k_1^{n+1}(t)\int_D u^{(n+1)(p-1)}\mathrm{d}x \leqslant Cn(p-1)\Phi^{1+\frac{1}{n}}(t)+\frac{1}{2\varepsilon_4}Cn(p-1)\Phi^{1+\frac{2}{n}}(t)+\frac{\varepsilon_4}{2}Cn(p-1)k_1^n(t)\int_D\left|\nabla u^{\frac{n(p-1)}{2}}\right|^2\mathrm{d}x \tag{30}$$

$$n(p-1)k_2^{n+1}(t)\int_D v^{(n+1)(p-1)}\mathrm{d}x \leqslant Cn(p-1)\Psi^{1+\frac{1}{n}}(t)+\frac{1}{2\theta_4}Cn(p-1)\Psi^{1+\frac{2}{n}}(t)+\frac{\theta_4}{2}Cn(p-1)k_2^n(t)\int_D\left|\nabla v^{\frac{n(p-1)}{2}}\right|^2\mathrm{d}x \tag{31}$$

其中

$$\varepsilon_4=\frac{2b_1[n(p-1)-1](n+1)}{Cn^2(p-1)(n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}p_2)}$$

$$\theta_4=\frac{2c_1[n(p-1)-1](n+1)}{Cn^2(p-1)(n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}p_1)}$$

第四步，将(23)，(24)，(26)，(27)，(30)，(31)式分别回代入(20)式中，并令

$$y_1=n\alpha_1+\frac{b_3n(p-1)}{2}|D|^{1-\frac{q_1}{n(p-1)}}k_1^{\frac{q_1}{1-p}}(t)k_3(t),\quad y_2=\frac{b_2n^2(p-1)}{\rho_0}|D|^{1+\frac{1-q_1}{n(p-1)}}k_1^{\frac{1-q_1}{p-1}}(t)k_3(t)$$

$$z_1=n\alpha_2+\frac{c_3n(p-1)}{2}|D|^{1-\frac{q_2}{n(p-1)}}k_2^{\frac{q_2}{1-p}}(t)k_4(t),\quad z_2=\frac{c_2n^2(p-1)}{\rho_0}|D|^{1+\frac{1-q_2}{n(p-1)}}k_2^{\frac{1-q_2}{p-1}}(t)k_4(t)$$

$$y_3=\frac{b_2d[n(p-1)-1]}{\rho_0\varepsilon_1}|D|^{1+\frac{1-q_1}{n(p-1)}}k_1^{\frac{1-q_1}{p-1}}(t)k_3(t),\quad z_3=\frac{c_2d[n(p-1)-1]}{\rho_0\theta_1}|D|^{1+\frac{1-q_2}{n(p-1)}}k_2^{\frac{1-q_2}{p-1}}(t)k_4(t)$$

$$y_4=\frac{b_3n(p-1)}{2\rho_0^2}\left(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\right)C^{\frac{2(l_1-1)}{p-1}}|D|^{1-\frac{q_1}{n(p-1)}}k_1^{\frac{2(1-l_1)-q_1}{p-1}}(t)k_3(t)$$

$$z_4=\frac{c_3n(p-1)}{2\rho_0^2}\left(n^2+\frac{[n(p-1)+l_2-1]^2d^2}{\theta_2}\right)C^{\frac{2(l_2-1)}{p-1}}|D|^{1-\frac{q_2}{n(p-1)}}k_2^{\frac{2(1-l_2)-q_2}{p-1}}(t)k_4(t)$$

$$y_5=\frac{b_3n(p-l_1)}{2\rho_0^2}\left(n^2+\frac{[n(p-1)+l_1-1]^2d^2}{\varepsilon_2}\right)\varepsilon_3^{\frac{1-l_1}{p-l_1}}C^{\frac{2(l_1-1)}{p-1}}|D|^{1-\frac{q_1}{n(p-1)}}k_1^{\frac{2(1-l_1)-q_1}{p-1}}(t)k_3(t)$$

$$z_5=\frac{c_3n(p-l_2)}{2\rho_0^2}\left(n^2+\frac{[n(p-1)+l_2-1]^2d^2}{\theta_2}\right)\theta_3^{\frac{1-l_2}{p-l_2}}C^{\frac{2(l_2-1)}{p-1}}|D|^{1-\frac{q_2}{n(p-1)}}k_2^{\frac{2(1-l_2)-q_2}{p-1}}(t)k_4(t)$$

$$y_6=\frac{n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2}{(n+1)(p-1)}Cn(p-1),\quad y_7=\frac{n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2}{2\varepsilon_4(n+1)(p-1)}Cn(p-1)$$

$$z_6=\frac{n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1}{(n+1)(p-1)}Cn(p-1),\quad z_7=\frac{n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1}{2\theta_4(n+1)(p-1)}Cn(p-1)$$

而后结合 M_3 的定义，可以得到

$$G'(t)\leqslant y_1\Phi(t)+y_2\Phi^{1+\frac{q_1-1}{n(p-1)}}(t)+y_3\Phi^{1+\frac{2(q_1-1)}{n(p-1)}}(t)+y_4\Phi^{1+\frac{2(l_1+q_1-1)}{n(p-1)}}(t)+y_5\Phi^{1+\frac{2(l_1+q_1-1)}{n(p-l_1)}}(t)+y_6\Phi^{1+\frac{1}{n}}(t)+y_7\Phi^{1+\frac{2}{n}}(t)+z_1\Psi(t)+z_2\Psi^{1+\frac{q_2-1}{n(p-1)}}(t)+z_3\Psi^{1+\frac{2(q_2-1)}{n(p-1)}}(t)+z_4\Psi^{1+\frac{2(l_2+q_2-1)}{n(p-1)}}(t)+z_5\Psi^{1+\frac{2(l_2+q_2-1)}{n(p-l_2)}}(t)+z_6\Psi^{1+\frac{1}{n}}(t)+z_7\Psi^{1+\frac{2}{n}}(t)+M_3 \tag{32}$$

又由 Young 不等式可知下述不等式成立

$$\begin{aligned}\Phi^{1+\beta}(t)&=(\Phi(t))^{1-\frac{\beta}{2}}(\Phi^3(t))^{\frac{\beta}{2}}\leqslant\left(1-\frac{\beta}{2}\right)\Phi(t)+\frac{\beta}{2}\Phi^3(t)\\ \Psi^{1+\eta}(t)&=(\Psi(t))^{1-\frac{\eta}{2}}(\Psi^3(t))^{\frac{\eta}{2}}\leqslant\left(1-\frac{\eta}{2}\right)\Psi(t)+\frac{\eta}{2}\Psi^3(t)\end{aligned} \tag{33}$$

将(33)式分别应用于(32)式的 $\Phi^{1+\frac{q_1-1}{n(p-1)}}(t)$，$\Psi^{1+\frac{q_2-1}{n(p-1)}}(t)$，$\Phi^{1+\frac{2(q_1-1)}{n(p-1)}}(t)$，$\Psi^{1+\frac{2(q_2-1)}{n(p-1)}}(t)$，$\Phi^{1+\frac{2(l_1+q_1-1)}{n(p-1)}}(t)$，$\Psi^{1+\frac{2(l_2+q_2-1)}{n(p-1)}}(t)$，$\Phi^{1+\frac{2(l_1+q_1-1)}{n(p-l_1)}}(t)$，$\Psi^{1+\frac{2(l_2+q_2-1)}{n(p-l_2)}}(t)$，$\Phi^{1+\frac{1}{n}}(t)$，$\Psi^{1+\frac{1}{n}}(t)$，$\Phi^{1+\frac{2}{n}}(t)$，$\Psi^{1+\frac{2}{n}}(t)$ 中可得

$$G'(t) \leqslant A_1(t)\Phi(t) + B_1(t)\Phi^3(t) + A_2(t)\Psi(t) + B_2(t)\Psi^3(t) + M_3 \tag{34}$$

其中：

$$\begin{aligned}
A_1(t) = {} & n\alpha_1 + \frac{b_3 n(p-1)}{2} |D|^{1-\frac{q_1}{n(p-1)}} k_1^{\frac{q_1}{1-p}}(t)k_3(t) + \frac{nb_2[2n(p-1)-q_1+1]}{2\rho_0} |D|^{1+\frac{1-q_1}{n(p-1)}} k_1^{\frac{1-q_1}{p-1}}(t)k_3(t) + \\
& \frac{b_2 d[n(p-1)-1][n(p-1)-q_1+1]}{n(p-1)\rho_0\varepsilon_1} |D|^{1+\frac{1-q_1}{n(p-1)}} k_1^{\frac{1-q_1}{p-1}}(t)k_3(t) + \\
& \frac{b_3[n(p-1)-q_1-l_1+1]}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_1-1]^2 d^2}{\varepsilon_2}\right)C^{\frac{2(l_1-1)}{p-1}} |D|^{1-\frac{q_1}{n(p-1)}} k_1^{\frac{2(1-l_1)-q_1}{p-1}}(t)k_3(t) + \\
& \frac{b_3[n(p-l_1)-q_1-l_1+1]}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_1-1]^2 d^2}{\varepsilon_2}\right)\varepsilon_3^{\frac{1-l_1}{p-l_1}} C^{\frac{2(l_1-1)}{p-1}} |D|^{1-\frac{q_1}{n(p-1)}} k_1^{\frac{2(1-l_1)-q_1}{p-1}}(t)k_3(t) + \\
& \frac{C(2n-1)[n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2]}{2(n+1)} + \frac{C(n-1)[n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2]}{2\varepsilon_4(n+1)}
\end{aligned}$$

$$\begin{aligned}
A_2(t) = {} & n\alpha_2 + \frac{c_3 n(p-1)}{2} |D|^{1-\frac{q_2}{n(p-1)}} k_2^{\frac{q_2}{1-p}}(t)k_4(t) + \frac{nc_2[2n(p-1)-q_2+1]}{2\rho_0} |D|^{1+\frac{1-q_2}{n(p-1)}} k_2^{\frac{1-q_2}{p-1}}(t)k_4(t) + \\
& \frac{c_2 d[n(p-1)-1][n(p-1)-q_2+1]}{n(p-1)\rho_0\theta_1} |D|^{1+\frac{1-q_2}{n(p-1)}} k_2^{\frac{1-q_2}{p-1}}(t)k_4(t) + \\
& \frac{c_3[n(p-1)-q_2-l_2+1]}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_2-1]^2 d^2}{\theta_2}\right)C^{\frac{2(l_2-1)}{p-1}} |D|^{1-\frac{q_2}{n(p-1)}} k_2^{\frac{2(1-l_2)-q_2}{p-1}}(t)k_4(t) + \\
& \frac{c_3[n(p-l_2)-q_2-l_2+1]}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_2-1]^2 d^2}{\theta_2}\right)\theta_3^{\frac{1-l_2}{p-l_2}} C^{\frac{2(l_2-1)}{p-1}} |D|^{1-\frac{q_2}{n(p-1)}} k_2^{\frac{2(1-l_2)-q_2}{p-1}}(t)k_4(t) + \\
& \frac{C(2n-1)[n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1]}{2(n+1)} + \frac{C(n-1)[n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1]}{2\theta_4(n+1)}
\end{aligned}$$

$$\begin{aligned}
B_1(t) = {} & \frac{nb_2(q_1-1)}{2\rho_0} |D|^{1+\frac{1-q_1}{n(p-1)}} k_1^{\frac{1-q_1}{p-1}}(t)k_3(t) + \frac{b_2 d[n(p-1)-1](q_1-1)}{n(p-1)\rho_0\varepsilon_1} |D|^{1+\frac{q_1}{n(p-1)}} k_1^{\frac{1-q_1}{p-1}}(t)k_3(t) + \\
& \frac{b_3[n(p-l_1)-q_1-l_1+1]}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_1-1]^2 d^2}{\varepsilon_2}\right)C^{\frac{2(l_1-1)}{p-1}} |D|^{1-\frac{q_1}{n(p-1)}} k_1^{\frac{2(1-l_1)-q_1}{p-1}}(t)k_3(t) + \\
& \frac{b_3(l_1+q_1-1)}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_1-1]^2 d^2}{\varepsilon_2}\right)\varepsilon_3^{\frac{1-l_1}{p-l_1}} C^{\frac{2(l_1-1)}{p-1}} |D|^{1-\frac{q_1}{n(p-1)}} k_1^{\frac{2(1-l_1)-q_1}{p-1}}(t)k_3(t) + \\
& \frac{C[n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2]}{2(n+1)} + \frac{C[n(p-1)-1+k_1^{-(n+1)}(t)k_2^{n+1}(t)p_2]}{2\varepsilon_4(n+1)}
\end{aligned}$$

$$\begin{aligned}
B_2(t) = {} & \frac{nc_2(q_2-1)}{2\rho_0} |D|^{1+\frac{1-q_2}{n(p-1)}} k_2^{\frac{1-q_2}{p-1}}(t)k_4(t) + \frac{c_2 d[n(p-1)-1](q_2-1)}{n(p-1)\rho_0\theta_1} |D|^{1+\frac{q_2}{n(p-1)}} k_2^{\frac{1-q_2}{p-1}}(t)k_4(t) + \\
& \frac{c_3[n(p-l_2)-q_2-l_2+1]}{2\rho_0^2}\left(n^2 + \frac{[n(p-1)+l_2-1]^2 d^2}{\theta_2}\right)C^{\frac{2(l_2-1)}{p-1}} |D|^{1-\frac{q_2}{n(p-1)}} k_2^{\frac{2(1-l_2)-q_2}{p-1}}(t)k_4(t) + \\
& \frac{c_3(l_1+q_2-1)}{2{\rho_0}^2}\left(n^2 + \frac{[n(p-1)+l_2-1]^2 d^2}{\theta_2}\right)\theta_3^{\frac{1-l_2}{p-l_2}} C^{\frac{2(l_2-1)}{p-1}} |D|^{1-\frac{q_2}{n(p-1)}} k_2^{\frac{2(1-l_2)-q_2}{p-1}}(t)k_4(t) + \\
& \frac{C[n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1]}{2(n+1)} + \frac{C[n(p-1)-1+k_2^{-(n+1)}(t)k_1^{n+1}(t)p_1]}{2\theta_4(n+1)}
\end{aligned}$$

由不等式 $\Phi^3(t) + \Psi^3(t) \leqslant (\Phi(t) + \Psi(t))^3$，$t \geqslant 0$ 及 M_1, M_2, M_3 的定义，我们有

$$G'(t) \leqslant M_1(\Phi(t) + \Psi(t)) + M_2(\Phi(t) + \Psi(t))^3 + M_3 = M_1 G(t) + M_2 G^3(t) + M_3 \tag{35}$$

现对(35) 式由 0 到 t 积分，通过极限 t 趋于 t^* 即得

$$t^* \geqslant \int_{G(0)}^{+\infty} \frac{d\tau}{M_1\tau + M_2\tau^3 + M_3} \tag{36}$$

定理 2 证明完毕.

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