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一类系数依赖于时间的非线性项的半线性双波动方程解的爆破研究①

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摘要:考虑了一类系数依赖于时间的非线性项的半线性双波动方程解的爆破情况.运用微分不等式方法和迭代方法证明了在次临界情况下半线性双波动方程柯西问题的解在有限时间内爆破,且给出了生命跨度的上界估计.进一步推广了波动方程在高阶上柯西问题的有关结果.

关 键 词:非线性项;半线性双波动方程;爆破

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Blow-up Study on Semi-linear Double-Wave Equation with Time-Dependent Coefficient on Nonlinearity

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Abstract: In this paper, the blow-up of solutions has been considered to a class of semi-linear double-wave equations with time-dependent coefficient on the nonlinearity. By means of differential inequalities and an iteration argument, the blow-up and the upper bound of the lifespan of solutions have been obtained to the Cauchy problem for semi-linear double-wave equations in the subcritical case, which generalize further the facts on the Cauchy problem for wave equations in high orders.

Key words: nonlinearity; semi-linear double-wave equation; blow-up

近年来,有关半线性波动方程柯西问题的研究受到广泛的关注.有很多学者[1-9]研究了如下波动方程的柯西问题

$$\begin{cases} u_{tt} - \Delta u = |u(t, x)|^p, (x, t) \in \mathbb{R}^n \times (0, T) \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), x \in \mathbb{R}^n \end{cases} \tag{1}$$

其中 $p > 1$,$n \geqslant 1$ 和 $u = u(t, x) \in \mathbb{R}$,$\varepsilon > 0$.

众所周知,(1) 式中的临界指数 $P_{crit}(n)$ 即 Strauss 指数在波动方程解的全局性与爆破研究中起着重要作用.(1) 式中的临界指数 $P_{crit}(n)$ 由下面一元二次方程的正根表示

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$$\frac{n-1}{2}p_{crit}^2-\frac{n-1}{2}P_{crit}(n)-1=0,\ n\geqslant 2$$

也就是

$$P_{crit}(n)=\frac{n+1+\sqrt{n^2+10n-7}}{2(p-1)}$$

对于 $n=1$，有 $P_{crit}(1)=\infty$.

对于(1)式的研究，学者们主要采用的方法是基于微分不等式和Kato引理. 然而，Kato引理只适用于二阶的微分方程，对于高阶的波动方程(比如四阶)，则需要寻找其他的办法. 近来有学者采用迭代办法研究了某些双曲方程解的全局性和爆破问题[10-16]. 有关其他的偏微分方程解的爆破问题研究可参考文献[17-19].

本文研究如下系数依赖于时间的非线性项的半线性双波动方程解的爆破问题

$$\begin{cases}(\partial_t^2-\Delta)^2u(t,x)=f(t)\,|u(t,x)|^p,\ (x,t)\in\mathbb{R}^n\times(0,T)\\(u,u_t,u_{tt},u_{ttt})(0,x)=\varepsilon(u_0,u_1,u_2,u_3)(x),\ x\in\mathbb{R}^n\end{cases}\tag{2}$$

其中 $f(t)=(1+t)^{-\alpha}$，$0<\alpha<2$，$p>1$，$\varepsilon>0$，Δ 是拉普拉斯算子.

目前，有关高阶的半线性双波动方程柯西问题解的爆破研究尚未得到展开. 其主要难点在于如何构造测试函数通过迭代方法来解决高阶波动方程柯西问题研究中出现的问题. 本文通过选取合适的测试函数进行迭代得到了在非临界情况下系数依赖于时间的非线性项的半线性双波动方程解的上界估计.

首先给出(2)式的柯西问题能量解的定义

定义1 设 $(u_0,u_1,u_2,u_3)\in H^3(\mathbb{R}^n)\times H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$，对于 $u\in C([0,T),H^3(\mathbb{R}^n))\cap C^1([0,T),H^2(\mathbb{R}^n))\cap C^2([0,T),H^1(\mathbb{R}^n))\cap C^3([0,T),L^2(\mathbb{R}^n))$ 且 $u\in L_{loc}^p([0,T)\times\mathbb{R}^n)$ 满足

$$\begin{aligned}&\int_{\mathbb{R}^n}u_{ttt}(t,x)\varphi(t,x)\mathrm{d}x-\int_0^t\int_{\mathbb{R}^n}u_{ttt}(s,x)\varphi_t(s,x)\mathrm{d}x\mathrm{d}s+\\&2\int_0^t\int_{\mathbb{R}^n}\nabla u_{tt}(s,x)\cdot\nabla\varphi(s,x)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}u(s,x)\Delta^2\varphi(s,x)\mathrm{d}x\mathrm{d}s=\\&\int_0^t\int_{\mathbb{R}^n}(1+s)^{-\alpha}\,|u(s,x)|^p\varphi(s,x)\mathrm{d}x\mathrm{d}s+\int_{\mathbb{R}^n}u_{ttt}(0,x)\varphi(0,x)\mathrm{d}x\end{aligned}\tag{3}$$

其中：$\varphi(t,x)\in C_0^\infty([0,T)\times\mathbb{R}^n)$ 和 $t\in[0,T)$.

对于(3)式，由分部积分可得

$$\begin{aligned}&\int_{\mathbb{R}^n}u_{ttt}(t,x)\varphi(t,x)\mathrm{d}x-\int_{\mathbb{R}^n}u_{tt}(t,x)\varphi_t(t,x)\mathrm{d}x+\int_{\mathbb{R}^n}u_t(t,x)\varphi_{tt}(t,x)\mathrm{d}x-\\&\int_{\mathbb{R}^n}u(t,x)\varphi_{ttt}(t,x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}^n}u(s,x)\cdot\varphi_{tttt}(s,x)\mathrm{d}x\mathrm{d}s-2\int_{\mathbb{R}^n}u_t(t,x)\Delta\varphi(t,x)\mathrm{d}x+\\&2\int_{\mathbb{R}^n}u(t,x)\Delta\varphi_t(t,x)\mathrm{d}x-2\int_0^t\int_{\mathbb{R}^n}u(s,x)\Delta\varphi_{tt}(s,x)\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{R}^n}\Delta\nabla u(s,x)\cdot\nabla\varphi(s,x)\mathrm{d}x\mathrm{d}s=\\&\int_0^t\int_{\mathbb{R}^n}(1+t)^{-\alpha}\,|u(s,x)|^p\varphi(s,x)\mathrm{d}x\mathrm{d}s+\varepsilon\int_{\mathbb{R}^n}u_3(x)\varphi(0,x)\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}u_2(x)\varphi_t(0,x)\mathrm{d}x+\\&\varepsilon\int_{\mathbb{R}^n}u_1(x)\varphi_{tt}(0,x)\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}u_0(x)\varphi_{ttt}(0,x)\mathrm{d}x-2\varepsilon\int_{\mathbb{R}^n}u_1(x)\Delta\varphi(0,x)\mathrm{d}x+\\&2\varepsilon\int_{\mathbb{R}^n}u_0(x)\Delta\varphi_t(0,x)\mathrm{d}x\end{aligned}\tag{4}$$

令 $t\to T$，则 u 满足(2)式定义的弱解的定义.

1 本文主要结果

定理1 设

$$1<p\begin{cases}<\infty,\ 如果\ n=1,2,3\\<p_0(n,\alpha),\ 如果\ n\geqslant 4\end{cases}$$

其中：$p_0(n,\alpha)=\dfrac{n+3-2\alpha+\sqrt{(n+3-2\alpha)^2+16n-48}}{2(n-3)}$，$(u_0,u_1,u_2,u_3)\in H^3(\mathbb{R}^n)\times H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ 是非负的紧致函数. 初始条件的支集落在半径为 R 的球B_R 内，使得 $u_i(i=0,1,2,3)$ 不恒为0. 特别的，假设$\int_{\mathbb{R}^n}(u_3(x)+u_2(x)-u_1(x)-u_0(x))\mathrm{d}x>0$ 和$\int_{\mathbb{R}^n}(2u_1(x)-u_0(x)-u_2(x))\mathrm{d}x>0$. 如果 u 是(2) 式的解，其生命跨度 $T(\varepsilon)$ 满足 $u(t,\cdot)\subset B_{t+R}$，$t\in(0,T)$，那么存在一个正常数 $\varepsilon_0=\varepsilon_0(u_0,u_1,u_2,u_3,n,p,\alpha,R)$，使得当 $\varepsilon\in(0,\varepsilon_0]$ 时，u 在有限时间爆破，其生命跨度的上界估计为

$$T(\varepsilon)\leqslant\widetilde{\widetilde{C}}\varepsilon^{-\frac{2p(p-1)}{Y(n,p,\alpha)}}$$

其中$\widetilde{\widetilde{C}}$ 独立于ε，且

$$Y(n,p,\alpha)=(n+3-2\alpha)p+4-(n-3)p^2 \tag{5}$$

2　爆破时间的上界估计

设

$$U(t)=\int_{\mathbb{R}^n}u(t,x)\mathrm{d}x \tag{6}$$

(4) 式中，取 $\varphi\equiv 1$，$\{(s,x)\in[0,t]\times\mathbb{R}^n:|x|\leqslant R+s\}$，可得

$$\int_{\mathbb{R}^n}u_{ttt}(t,x)\mathrm{d}x=\int_0^t\int_{\mathbb{R}^n}(1+s)^{\alpha}|u(s,x)|^p\mathrm{d}x\mathrm{d}s+\varepsilon\int_{\mathbb{R}^n}u_3(x)\mathrm{d}x \tag{7}$$

联立(6),(7) 式，得到

$$U'''(t)=U'''(0)+\int_0^t\int_{\mathbb{R}^n}(1+s)^{-\alpha}|u(s,x)|^p\mathrm{d}x\mathrm{d}s \tag{8}$$

对(8) 式关于 t 积分 3 次，可得

$$\begin{aligned}U(t)=&U(0)+U'(0)t+\frac{1}{2}U''(0)t^2+\frac{1}{6}U'''(0)t^3+\\&\int_0^t\int_0^s\int_0^\tau\int_0^\sigma\int_{\mathbb{R}^n}(1+\eta)^{-\alpha}|u(\eta,x)|^p\mathrm{d}x\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant 0\end{aligned} \tag{9}$$

因为支集 $u(t,\cdot)\subset B_{t+R}$，$\forall t\in(0,T)$，由 Hölder 不等式，可得

$$\int_{\mathbb{R}^n}|u(\eta,x)|^p\mathrm{d}x\geqslant C(R+\eta)^{-n(p-1)}(U(\eta))^p \tag{10}$$

由(9),(10) 式，可得

$$\begin{aligned}U(t)\geqslant&C\int_0^t\int_0^s\int_0^\tau\int_0^\sigma(1+\eta)^{-\alpha}(R+\eta)^{-n(p-1)}(U(\eta))^p\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\&C\int_0^t\int_0^s\int_0^\tau\int_0^\sigma(R+\eta)^{-[\alpha+n(p-1)]}(U(\eta))^p\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\end{aligned} \tag{11}$$

下面将通过对 $U(t)$ 的下界进行迭代完成定理的证明. (11) 式确定了迭代的框架. 为了推导 $U(t)$ 的第一个下界估计，引入如下函数[20]

$$\Phi(x)=\begin{cases}\mathrm{e}^x+\mathrm{e}^{-x},\ n=1\\ \int_{S^{n-1}}\mathrm{e}^{x\cdot\omega}\mathrm{d}\sigma_\omega,\ n\geqslant 2\end{cases}$$

函数 $\Phi(x)$ 是正的，并且有下面的性质

$$\Delta\Phi(x)=\Phi(x),\ \Phi(x)\sim|x|^{-\frac{n-1}{2}}\mathrm{e}^{|x|},\ \text{当}\ |x|\to\infty\ \text{时}$$

令 $\Psi=\Psi(t,x)=\mathrm{e}^{-t}\Phi(x)$，显然，$\Psi$ 满足$(\partial_t^2-\Delta)^2\Psi=0$.

定义辅助函数

$$U_0(t)=\int_{\mathbb{R}^n}u(t,x)\Psi(t,x)\mathrm{d}x \tag{12}$$

对(8) 式关于时间 t 求导数，得

$$U''''(t)=\int_{\mathbb{R}^n}(1+t)^{-\alpha}|u(t,x)|^p\mathrm{d}x=(1+t)^{-\alpha}\int_{\mathbb{R}^n}|u(t,x)|^p\mathrm{d}x \tag{13}$$

应用 Hölder 不等式于(12)式,得到

$$\int_{\mathbb{R}^n}|u(s,x)|^p\mathrm{d}x\geqslant|U_0(s)|^p\left(\int_{B_{R+s}}|\Psi(s,x)|^{\frac{p}{p-1}}\mathrm{d}x\right)^{-(p-1)} \tag{14}$$

将测试函数 Ψ 应用到(3)式,有

$$\begin{aligned}&\int_{\mathbb{R}^n}u_{ttt}(t,x)\Psi(t,x)\mathrm{d}x-\int_0^t\int_{\mathbb{R}^n}u_{ttt}(s,x)\Psi_t(s,x)\mathrm{d}x\mathrm{d}s+\\&2\int_0^t\int_{\mathbb{R}^n}\nabla u_{tt}(s,x)\cdot\nabla\Psi(s,x)\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{R}^n}\Delta\nabla u(s,x)\cdot\nabla\Psi(s,x)\mathrm{d}x\mathrm{d}s=\\&\int_0^t\int_{\mathbb{R}^n}(1+s)^{-\alpha}|u(s,x)|^p\Psi(s,x)\mathrm{d}x\mathrm{d}s+\int_{\mathbb{R}^n}u_{ttt}(0,x)\Psi(0,x)\mathrm{d}x\end{aligned} \tag{15}$$

对(15)式分部积分并注意到 Ψ 的性质,可得

$$\begin{aligned}&\int_{\mathbb{R}^n}u_{ttt}(t,x)\Psi(t,x)\mathrm{d}x+\int_{\mathbb{R}^n}u_{tt}(t,x)\Psi(t,x)\mathrm{d}x-\int_{\mathbb{R}^n}u_t(t,x)\Psi(t,x)\mathrm{d}x-\\&\int_{\mathbb{R}^n}u(t,x)\Psi(t,x)\mathrm{d}x=\varepsilon I[u_0,u_1,u_2,u_3]+\int_0^t\int_{\mathbb{R}^n}(1+s)^{-\alpha}|u(s,x)|^p\Psi(s,x)\mathrm{d}x\mathrm{d}s\end{aligned} \tag{16}$$

其中

$$I[u_0,u_1,u_2,u_3]=\int_{\mathbb{R}^n}(u_3(x)+u_2(x)-u_1(x)-u_0(x))\Phi(x)\mathrm{d}x$$

联立(12)式和(16)式,得

$$\begin{aligned}&U'''_0(t)+4U''_0(t)+4U'_0(t)=\varepsilon I[u_0,u_1,u_2,u_3]+\int_0^t\int_{\mathbb{R}^n}(1+s)^{-\alpha}|u(s,x)|^p\Psi(s,x)\mathrm{d}x\mathrm{d}s\geqslant\\&\varepsilon I[u_0,u_1,u_2,u_3]\end{aligned} \tag{17}$$

设

$$F(t)=U''_0(t)+2U'_0(t)$$

于是,(17)式可化为

$$F'(t)+2F(t)\geqslant\varepsilon I[u_0,u_1,u_2,u_3] \tag{18}$$

对(18)式积分,得

$$F(t)\geqslant\left(F(0)-\frac{1}{2}\varepsilon I[u_0,u_1,u_2,u_3]\right)\mathrm{e}^{-2t}+\frac{1}{2}\varepsilon I[u_0,u_1,u_2,u_3] \tag{19}$$

由(19)式和 $F(t)$ 的定义,有

$$U''_0(t)+2U'_0(t)\geqslant\left(F(0)-\frac{1}{2}\varepsilon I[u_0,u_1,u_2,u_3]\right)\mathrm{e}^{-2t}+\frac{1}{2}\varepsilon I[u_0,u_1,u_2,u_3] \tag{20}$$

对(20)式关于 t 求积分,可推出

$$\begin{aligned}U_0(t)\geqslant&\int_{\mathbb{R}^n}u_0(x)\Phi(x)\mathrm{d}x+\frac{\varepsilon}{4}(1-\mathrm{e}^{-2t})\int_{\mathbb{R}^n}(3u_1(x)-2u_0(x)-u_3(x))\Phi(x)\mathrm{d}x+\\&\frac{\varepsilon}{4}t\int_{\mathbb{R}^n}(u_3(x)+u_2(x)-u_1(x)-u_0(x))\Phi(x)\mathrm{d}x+\\&\frac{\varepsilon}{4}t\,\mathrm{e}^{-2t}\int_{\mathbb{R}^n}(u_3(x)+u_0(x)-u_1(x)-u_2(x))\Phi(x)\mathrm{d}x\geqslant\\&\int_{\mathbb{R}^n}u_0(x)\Phi(x)\mathrm{d}x+\frac{\varepsilon\delta}{4}\int_{\mathbb{R}^n}(2u_1(x)-u_0(x)-u_2(x))\Phi(x)\mathrm{d}x+\\&\frac{\varepsilon}{4}t\int_{\mathbb{R}^n}(u_3(x)+u_2(x)-u_1(x)-u_0(x))\Phi(x)\mathrm{d}x\end{aligned} \tag{21}$$

其中 $\delta=\min\{1-\mathrm{e}^{-2t},t\,\mathrm{e}^{-2t}\}$.

由定理的条件,可得当 $t\geqslant t_0$ 时,有

$$U_0(t)\geqslant\widetilde{C}\varepsilon t \tag{22}$$

由 Ψ 的渐近性，可得

$$\int_{B_{R+s}} |\Psi(s,x)|^{\frac{p}{p-1}}\mathrm{d}x \leqslant \widetilde{K}(R+s)^{(n-1)(1-\frac{p'}{2})} \tag{23}$$

其中 $\widetilde{K}$ 为正常数，p' 为 p 的共轭指数.

由(14),(22) 和(23) 式有

$$\int_{\mathbb{R}^n} |u(s,x)|^p\mathrm{d}x \geqslant C_0\varepsilon^p(R+s)^{(n-1)-\frac{(n-1)p}{2}}s^p \tag{24}$$

其中$C_0=\widetilde{C}^p\widetilde{K}^{-(p-1)}$，$s\geqslant t_0$.

联立(13) 和(24) 式可得

$$U''''(t)\geqslant C_0\varepsilon^p(R+t)^{(n-1)-\left[\frac{(n-1)p}{2}+\alpha\right]}t^p \tag{25}$$

其中 $t\geqslant t_0$.

对(25) 式求积分，有

$$\begin{aligned}U(t)\geqslant\ & U(t_0)+U'(t_0)(t-t_0)+U''(t_0)\frac{(t-t_0)^2}{2}+U'''(t_0)\frac{(t-t_0)^3}{6}+\\ & C_0\varepsilon^p\int_{t_0}^t\int_{t_0}^s\int_{t_0}^\tau\int_{t_0}^\sigma(R+\eta)^{(n-1)-\left[\frac{(n-1)p}{2}+\alpha\right]}\eta^p\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\ & \frac{C_0\varepsilon^p}{(n+p)(n+p+1)(n+p+2)(n+p+3)}(R+t)^{-\left[\frac{(n-1)p}{2}+\alpha\right]}(t-t_0)^{n+p+3}\end{aligned} \tag{26}$$

(26) 式可记为

$$U(t)\geqslant K_0(R+t)^{-\alpha_0}(t-t_0)^{\beta_0} \tag{27}$$

其中 $K_0=\dfrac{C_0\varepsilon^p}{(n+p)(n+p+1)(n+p+2)(n+p+3)}$，$\alpha_0=\dfrac{(n-1)p}{2}+\alpha$，$\beta_0=n+p+3$，$t\geqslant t_0$.

接下来，将通过迭代来推导 $U(t)$ 的下界

$$U(t)\geqslant K_j(R+t)^{-\alpha_j}(t-t_0)^{\beta_j} \tag{28}$$

其中非负实序列$\{K_j\}_{j\in\mathbb{N}}$，$\{\alpha_j\}_{j\in\mathbb{N}}$，$\{\beta_j\}_{j\in\mathbb{N}}$ 将在下文定义.

联立(11) 和(28) 式，得

$$\begin{aligned}U(t)\geqslant\ & CK_j^p\int_{t_0}^t\int_{t_0}^s\int_{t_0}^\tau\int_{t_0}^\sigma(R+\eta)^{-[\alpha+n(p-1)]-\alpha_j p}(\eta-t_0)^{p\beta_j}\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}s\geqslant\\ & \frac{CK_j^p}{(p\beta_j+1)(p\beta_j+2)(p\beta_j+3)(p\beta_j+4)(p\beta_j+5)}(R+t)^{-[\alpha+n(p-1)]-\alpha_j p}(t-t_0)^{p\beta_j+5}\end{aligned} \tag{29}$$

接着取

$$K_{j+1}=\frac{CK_j^p}{(p\beta_j+1)(p\beta_j+2)(p\beta_j+3)(p\beta_j+4)(p\beta_j+5)},\ \alpha_{j+1}=\alpha+n(p-1)+\alpha_j p,\ \beta_{j+1}=p\beta_j+5 \tag{30}$$

则(29) 式可化为

$$U(t)\geqslant K_{j+1}(R+t)^{-\alpha_{j+1}}(t-t_0)^{\beta_{j+1}} \tag{31}$$

(31) 式表明(28) 式对于 $j+1$ 是成立的. 接下来，将对 K_j,α_j,β_j 进行估计.

由(30) 式有

$$\begin{aligned}\alpha_j &=(\alpha+n(p-1))(1+p+p^2+\cdots p^{j-1})+\alpha_0p^j=\\ &\left(\frac{\alpha}{p-1}+n+\alpha_0\right)p^j-\left(\frac{\alpha}{p-1}+n\right)\\ \beta_j &=5(1+p+p^2+\cdots p^{j-1})+\beta_0p^j=\\ &\left(\frac{5}{p-1}+\beta_0\right)p^j-\frac{5}{p-1}\end{aligned} \tag{32}$$

又由于

$$(p\beta_{j-1}+1)(p\beta_{j-1}+2)(p\beta_{j-1}+3)(p\beta_{j-1}+4)(p\beta_{j-1}+5)\leqslant(p\beta_{j-1}+5)^5=\beta_j^5\leqslant$$

$$\left(\frac{5}{p-1}+\beta_0\right)^5 p^{5j} \tag{33}$$

联立(30)和(33)式，得到

$$K_j \geqslant C\left(\frac{5}{p-1}+\beta_0\right)^{-5} p^{-5j} K_{j-1}^p = D p^{-5j} K_{j-1}^p \tag{34}$$

其中 $D = C\left(\frac{5}{p-1}+\beta_0\right)^{-5}$.

对(34)式两边取对数可得

$$\log K_j \geqslant p^j\left(\log K_0 - \frac{5p\log p}{(p-1)^2} + \frac{\log D}{p-1}\right) + \frac{5j\log p}{p-1} + \frac{5p\log p}{(p-1)^2} - \frac{\log D}{p-1},\ \forall j \in \mathbb{N} \tag{35}$$

令 $j_0 = j_0(n, p) \in \mathbb{N}$ 为满足

$$j_0 \geqslant \frac{\log D}{5\log p} - \frac{p}{p-1}$$

的最小正整数，从而，对于 $j \geqslant j_0$，由(35)式可得

$$\log K_j \geqslant p^j\left(\log K_0 - \frac{5p\log p}{(p-1)^2} + \frac{\log D}{p-1}\right) = p^j \log(E_0 \varepsilon^p) \tag{36}$$

其中 $E_0 = E_0(n, p) > 0$.

联立(28),(32)和(36)式，得到

$$\begin{aligned} U(t) \geqslant\ & \mathrm{e}^{p^j \log(E_0\varepsilon^p)} (R+t)^{-\left(\frac{\alpha}{p-1}+n+\alpha_0\right)p^j + \left(\frac{\alpha}{p-1}+n\right)} (t-t_0)^{\left(\frac{5}{p-1}+\beta_0\right)p^j - \frac{5}{p-1}} = \\ & \mathrm{e}^{p^j\left(\log(E_0\varepsilon^p) - \left(\frac{\alpha}{p-1}+n+\alpha_0\right)\log(R+t) + \left(\frac{5}{p-1}+\beta_0\right)\log(t-t_0)\right)} \times \\ & (R+t)^{\frac{\alpha}{p-1}+n} (t-t_0)^{-\frac{5}{p-1}} \end{aligned} \tag{37}$$

其中 $j \geqslant j_0$，$t \geqslant t_0$.

当 $t \geqslant R + 2t_0$ 时，有 $\log(R+t) \leqslant \log(2(t-t_0))$. 于是(37)式化为

$$U(t) \geqslant \mathrm{e}^{p^j\left(\log\left(E_0\varepsilon^p 2^{-\left(\frac{\alpha}{p-1}+n+\alpha_0\right)} (t-t_0)^{\frac{5}{p-1}+\beta_0-\left(\frac{\alpha}{p-1}+n+\alpha_0\right)}\right)\right)} \tag{38}$$

其中 $t - t_0$ 的指数为

$$\frac{5}{p-1}+\beta_0-\left(\frac{\alpha}{p-1}+n+\alpha_0\right) = \frac{(n+3-2\alpha)p+4-(n-3)p^2}{2(p-1)} = \frac{\Upsilon(n, p, \alpha)}{2(p-1)} \tag{39}$$

由于 $0 < \alpha < 2$，当 $n = 1,2,3$ 时，$p > 1$；当 $n \geqslant 4$ 时，$1 < p < p_0(n, \alpha)$.

取 $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, u_3, n, p, \alpha, R) > 0$，使得

$$\varepsilon_0^{-\frac{2p(p-1)}{\Upsilon(n, p, \alpha)}} \geqslant E_1 R$$

其中 $E_1 = \left(2^{-\left(\frac{\alpha}{p-1}+n+\alpha_0\right)}\varepsilon\right)^{-\frac{2p(p-1)}{\Upsilon(n, p, \alpha)}}$.

因此，当 $\varepsilon \in (0, \varepsilon_0]$ 和 $t - t_0 > E_1^{-1}\varepsilon^{-\frac{2p(p-1)}{\Upsilon(n, p, \alpha)}} \geqslant R$ 时，可得

$$\log\left(\varepsilon^p\, 2^{-\left(\frac{\alpha}{p-1}+n+\alpha_0\right)} E_0 (t-t_0)^{\frac{\Upsilon(n, p, \alpha)}{2p(p-1)}}\right) > 0$$

在(38)式中，令 $j \to \infty$，可推出当 $\varepsilon \in (0, \varepsilon_0]$ 和 $t - t_0 > E_1^{-1}\varepsilon^{-\frac{2p(p-1)}{\Upsilon(n, p, \alpha)}} \geqslant R$ 时 $U(t)$ 的下界爆破. 这表明方程(2)不存在全局解. 进一步可得 u 的局部的生命跨度估计

$$T(\varepsilon) \leqslant \widetilde{\widetilde{C}}\, \varepsilon^{-\frac{2p(p-1)}{\Upsilon(n, p, \alpha)}}$$

从而证明了定理 1.

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