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重尾指数估计量及其伪估计量的渐近关系①

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摘要：将重尾指数估计量的随机门限替换为非随机门限，得到了伪估计量，然后建立了原始估计量和伪估计量之间的渐近关系.

关 键 词：重尾指数量；伪估计量；二阶正规变化函数

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On Asymptotic Relationship Between Heavy-Tailed Index Estimator and Its Pseudo-Estimator

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Abstract: As an important research object of extreme value theory, extreme value index is often used to describe the thick tailed degree of distribution. Therefore, the estimation of extreme value index has attracted the attention of many statisticians, and is widely used in insurance, actuarial, finance and other fields. In this paper, the pseudo-estimator has been obtained by replacing the random threshold of the heavy tailed index estimator with the non-random threshold, and then the asymptotic relationship between the original estimator and the pseudo-estimator been established.

Key words: heavy tailed index estimator; pseudo-estimator; second-order regularly varying function

设$\{X_n, n \geqslant 1\}$为独立同分布随机变量序列，公共分布函数$F(x)$为重尾分布，即$\bar{F} \in RV_{-\frac{1}{\gamma}}$，其中重尾指数$\gamma > 0$，$\bar{F}(x) = 1 - F(x)$. 分布函数$F(x)$未知时，文献[1]提出了极值指数的 Hill 估计量. 在此基础上文献[2-3]进一步证明了 Hill 估计量$\sqrt{k}$阶渐近正态性. 作为 Hill 估计量的一个推广，文献[4-5]提出

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了如下定义的半参数估计量

$$\hat{\gamma}_n(k):=\sqrt{\frac{1}{2k}\sum_{i=1}^{k}(\log X_{n-i+1,n}-\log X_{n-k,n})^2}$$

其中$\{X_{i,n},1\leqslant i\leqslant n\}$为其升序统计量. 有关极值指数估计量及其应用的更多研究见文献[6-14].

本文受文献[7]构造伪估计量的启发，将半参数估计量$\hat{\gamma}_n(k)$的$X_{n-k,n}$替换为非随机的分位数$U\left(\frac{n}{k}\right)$，得到如下伪估计量

$$\tilde{\gamma}_n(k)=\frac{\sqrt{\sum_{i=1}^{n}\left(\log X_i-\log U\left(\frac{n}{k}\right)\right)^2\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}}{\sqrt{2\sum_{i=1}^{n}\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}}$$

其中$\mathbb{1}_{\{\cdot\}}$为示性函数. 本文将利用Lyapunov中心极限定理[2]和Cramér-Wold设计[15]，建立估计量$\hat{\gamma}_n(k)$与伪估计量$\tilde{\gamma}_n(k)$的渐近关系. 本文假定$U(t)=\left(\frac{1}{1-F}\right)^{\leftarrow}(t)$为二阶正规变化函数，即存在辅助函数$A(t)$，使得

$$\lim_{t\to\infty}\frac{1}{A(t)}\left(\frac{U(tx)}{U(t)}-x^{\gamma}\right)=x^{\gamma}\frac{x^{\rho}-1}{\rho}\tag{1}$$

对所有$x>0$成立. 显然，$A(t)\in RV_{\rho}$，$\rho\leqslant 0$.

对连续可微单增函数$f(x)$，定义

$$AE_f(t)=\sqrt{\frac{E((f(X)-f(t))^2\mathbb{1}_{\{X>t\}})}{2\bar{F}(t)}}$$

$$\hat{AE}_f(t)=\sqrt{\frac{n^{-1}\sum_{i=1}^{n}(f(X_i)-f(t))^2\mathbb{1}_{\{X_i>t\}}}{2\bar{F}_n(t)}}$$

其中$\bar{F}_n(t)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>t\}}$. 特别地，当$f(x)=\log x$时，有$\hat{AE}_{\log}(X_{n-k,n})=\hat{\gamma}_n(k)$和$\hat{AE}_{\log}\left(U\left(\frac{n}{k}\right)\right)=\tilde{\gamma}_n(k)$. 下面给出本文的主要结果，即$\hat{AE}_f(X_{n-k,n})$和$\hat{AE}_f\left(U\left(\frac{n}{k}\right)\right)$之间的渐近关系.

定理 1　在条件(1)下，假设序列k满足$k=k(n)\to\infty$，$\frac{k}{n}\to 0$和$\sqrt{k}A\left(\frac{n}{k}\right)=O(1)(n\to\infty)$. 函数$f$满足$f'\in RV_{a-1}$，且$0\leqslant 4a\gamma<1$. 则

$$\frac{(\hat{AE}_f(X_{n-k,n}))^2}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}-1=\left(\frac{\left(\hat{AE}_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}-1\right)-\left(\frac{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}{\bar{F}\left(U\left(\frac{n}{k}\right)\right)}-1\right)+o_P\left(\frac{1}{\sqrt{k}}\right)\tag{2}$$

特别地，令$f(x)=\log x$，可以得到如下推论.

推论 1　在条件(1)下，假设序列k满足$k=k(n)\to\infty$，$\frac{k}{n}\to 0$和$\sqrt{k}A\left(\frac{n}{k}\right)=O(1)(n\to\infty)$，则有

$$\hat{\gamma}_n(k)=\tilde{\gamma}_n(k)+o_P\left(\frac{1}{\sqrt{k}}\right)$$

引理 1　假设条件(1)成立. 函数f满足$f'\in RV_{a-1}$，且$0\leqslant 4a\gamma<1$. 当$t\to\infty$时，有

$$\frac{(AE_f(t))^2}{(tf'(t))^2}\to\frac{\gamma^2}{(1-a\gamma)(1-2a\gamma)},\quad\frac{(AE_f(t))^2}{tf'(t)f(t)}\to\frac{a\gamma^2}{(1-a\gamma)(1-2a\gamma)}\tag{3}$$

证 使用分部积分法，可得

$$(AE_f(t))^2=\frac{1}{\bar{F}(t)}\int_t^{\infty}\bar{F}(v)f'(v)(f(v)-f(t))\mathrm{d}v=\int_1^{\infty}\frac{\bar{F}(tw)}{\bar{F}(t)}tf'(tw)(f(tw)-f(t))\mathrm{d}w$$

再根据文献[2] 的命题 B.1.10，当 t 充分大时，对任意的 $\delta>0$ 有

$$\left|\frac{f(tw)-f(t)}{tf'(t)}-\frac{w^a-1}{a}\right|\leqslant\int_1^w\left|\frac{f'(tu)}{f'(t)}-u^{a-1}\right|\mathrm{d}u\leqslant\varepsilon w^{a+\delta}\tag{4}$$

加之 $\bar{F}\in RV_{-\frac{1}{\gamma}}$ 和 $f\in RV_a$，得

$$\lim_{t\to\infty}\frac{(AE_f(t))^2}{(tf'(t))^2}=\lim_{t\to\infty}\int_1^{\infty}\frac{\bar{F}(tw)f'(tw)}{\bar{F}(t)f'(t)}\frac{f(tw)-f(t)}{tf'(t)}\mathrm{d}w=\frac{\gamma^2}{(1-a\gamma)(1-2a\gamma)}$$

同理，

$$\lim_{t\to\infty}\frac{(AE_f(t))^2}{tf'(t)f(t)}=\lim_{t\to\infty}\int_1^{\infty}\frac{\bar{F}(tw)f'(tw)}{\bar{F}(t)f'(t)}\frac{f(tw)-f(t)}{f(t)}\mathrm{d}w=\frac{a\gamma^2}{(1-a\gamma)(1-2a\gamma)}$$

引理 1 得证.

引理 2 在条件(1) 下，假设序列 k 满足 $k=k(n)\to\infty$，$\frac{k}{n}\to 0$ 和$\sqrt{k}A\left(\frac{n}{k}\right)=O(1)(n\to\infty)$. 函数 f 满足 $f'\in RV_{a-1}$，且 $0\leqslant 4a\gamma<1$. 令

$$Z_{i,n}^{(f)}:=\frac{\sqrt{k}}{n}\left(\frac{\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}{E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X>U\left(\frac{n}{k}\right)\right\}}\right)}-1\right)$$

则当 $n\to\infty$ 时，有

$$\sum_{i=1}^n\mathrm{Var}(Z_{i,n}^{(f)})\to h(a,\gamma),\quad\sum_{i=1}^n E\mid Z_{i,n}^{(f)}\mid^{2+\delta}=O(k^{-\frac{\delta}{2}}2^{-2-\delta})\to 0\tag{5}$$

其中 $h(a,\gamma):=-\frac{6(1-a\gamma)(1-2a\gamma)}{(4a\gamma-1)(3a\gamma-1)}-1$，$\delta>0$. 特别地，$\sum_{i=1}^n Z_{i,n}^{(f)}\xrightarrow{d}N(0,h(a,\gamma))$.

证 根据$\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2=\frac{E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X>U\left(\frac{n}{k}\right)\right\}}\right)}{2\bar{F}\left(U\left(\frac{n}{k}\right)\right)}$ 和 $\bar{F}\left(U\left(\frac{n}{k}\right)\right)=\frac{k}{n}$，可得

$$\sum_{i=1}^n\mathrm{Var}(Z_{i,n}^{(f)})=\frac{E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^4\mid X>U\left(\frac{n}{k}\right)\right)}{4\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^4}-1$$

使用分部积分法，可得

$$E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^4\mid X>U\left(\frac{n}{k}\right)\right)=$$

$$\frac{4U\left(\frac{n}{k}\right)}{\bar{F}\left(U\left(\frac{n}{k}\right)\right)}\int_1^{\infty}f'\left(U\left(\frac{n}{k}\right)w\right)\bar{F}\left(U\left(\frac{n}{k}\right)w\right)\left(f\left(U\left(\frac{n}{k}\right)\right)-f\left(U\left(\frac{n}{k}\right)w\right)\right)^3\mathrm{d}w$$

加之 $\bar{F}\in RV_{-\frac{1}{\gamma}}$，$f\in RV_a$ 和(4) 式，当 $n\to\infty$ 时，可得

$$\frac{E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^4 \mid X>U\left(\frac{n}{k}\right)\right)}{\left(U\left(\frac{n}{k}\right)f'\left(U\left(\frac{n}{k}\right)\right)\right)^4}\to$$

$$4\int_1^{\infty} w^{a-\frac{1}{\gamma}-1}\cdot\left(\frac{1-w^a}{a}\right)^3\mathrm{d}w=-\frac{24\gamma^4}{(4a\gamma-1)(3a\gamma-1)(2a\gamma-1)(a\gamma-1)}$$

再结合引理 1，可得

$$\sum_{i=1}^{n}\mathrm{Var}(Z_{i,n}^{(f)})\to-\frac{6(1-a\gamma)(1-2a\gamma)}{(4a\gamma-1)(3a\gamma-1)}-1=:h(a,\gamma)$$

此外，当 n 充分大时，对任意的 $\delta>0$ 有

$$\sum_{i=1}^{n}E\mid Z_{i,n}^{(f)}\mid^{2+\delta}=\sum_{i=1}^{n}E\left|\frac{\sqrt{k}}{n}\left(\frac{\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}{E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X>U\left(\frac{n}{k}\right)\right\}}\right)}-1\right)\right|^{2+\delta}=$$

$$O\left(k^{-\frac{\delta}{2}}\left(\frac{E\left(\mid f(X)-f\left(U\left(\frac{n}{k}\right)\right)\mid^{4+2\delta}\mid X>U\left(\frac{n}{k}\right)\right)}{2^{2+\delta}\mid \mathrm{AE}_f\left(U\left(\frac{n}{k}\right)\right)\mid^{4+2\delta}}\right)\right)=$$

$$O(k^{-\frac{\delta}{2}}\cdot 2^{-2-\delta})=$$

$$o(1)$$

最后，根据 Lyapunov 中心极限定理可知 $\sum_{i=1}^{n}Z_{i,n}^{(f)}$ 收敛. 引理 2 得证.

定理 1 的证明　使用分部积分法，可得

$$\hat{AE}_f(X_{n-k,n})=\sqrt{\frac{1}{\bar{F}_n(X_{n-k,n})}\int_{X_{n-k,n}}^{\infty}\bar{F}_n(v)f'(v)(f(v)-f(X_{n-k,n}))\mathrm{d}v}\tag{6}$$

$$\hat{AE}_f\left(U\left(\frac{n}{k}\right)\right)=\sqrt{\frac{1}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\int_{U\left(\frac{n}{k}\right)}^{\infty}\bar{F}_n(v)f'(v)\left(f(v)-f\left(U\left(\frac{n}{k}\right)\right)\right)\mathrm{d}v}\tag{7}$$

将(6) 式与(7) 式分别平方后再相减，可得

$$(\hat{AE}_f(X_{n-k,n}))^2-\left(\hat{AE}_f\left(U\left(\frac{n}{k}\right)\right)\right)^2=$$

$$\frac{1}{\bar{F}_n(X_{n-k,n})}\int_{X_{n-k,n}}^{U\left(\frac{n}{k}\right)}\bar{F}_n(v)f'(v)f(v)\mathrm{d}v+\left(\frac{1}{\bar{F}_n(X_{n-k,n})}-\frac{1}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\right)\int_{U\left(\frac{n}{k}\right)}^{\infty}\bar{F}_n(v)f'(v)f(v)\mathrm{d}v+$$

$$\frac{f\left(U\left(\frac{n}{k}\right)\right)}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\int_{U\left(\frac{n}{k}\right)}^{X_{n-k,n}}\bar{F}_n(v)f'(v)\mathrm{d}v+\left(\frac{f\left(U\left(\frac{n}{k}\right)\right)}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}-\frac{f(X_{n-k,n})}{\bar{F}_n(X_{n-k,n})}\right)\int_{X_{n-k,n}}^{\infty}\bar{F}_n(v)f'(v)\mathrm{d}v\tag{8}$$

首先考虑等式(8) 右边的第一项与第三项. 令 $v=xU\left(\frac{n}{k}\right)$ 再结合 $\frac{1}{\bar{F}_n(X_{n-k,n})}=\frac{n}{k}$，可得

$$\frac{1}{\bar{F}_n(X_{n-k,n})}\int_{X_{n-k,n}}^{U\left(\frac{n}{k}\right)}\bar{F}_n(v)f'(v)f(v)\mathrm{d}v+\frac{f\left(U\left(\frac{n}{k}\right)\right)}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\int_{U\left(\frac{n}{k}\right)}^{X_{n-k,n}}\bar{F}_n(v)f'(v)\mathrm{d}v=$$

$$U\left(\frac{n}{k}\right)f'\left(U\left(\frac{n}{k}\right)\right)f\left(U\left(\frac{n}{k}\right)\right)\int_{\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}}^{1}\frac{n}{k}\bar{F}_n\left(xU\left(\frac{n}{k}\right)\right)\frac{f'\left(xU\left(\frac{n}{k}\right)\right)f\left(xU\left(\frac{n}{k}\right)\right)}{f'\left(U\left(\frac{n}{k}\right)\right)f\left(U\left(\frac{n}{k}\right)\right)}\mathrm{d}x+$$

$$U\left(\frac{n}{k}\right)f'\left(U\left(\frac{n}{k}\right)\right)f\left(U\left(\frac{n}{k}\right)\right)\int_{1}^{\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}}\frac{n}{k}\bar{F}_n\left(xU\left(\frac{n}{k}\right)\right)\frac{f'\left(xU\left(\frac{n}{k}\right)\right)}{f'\left(U\left(\frac{n}{k}\right)\right)}\mathrm{d}x$$

由文献[2]定理2.4.1知，$\sqrt{k}\left(\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}-1\right)\xrightarrow{d}N(0,1)$. 根据 $f'\in RV_{\alpha-1}$，且 $\bar{F}_n(X_{n-k,n})=\frac{k}{n}$，当 n 充分大时，可得

$$\int_{\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}}^{1}\frac{n}{k}\bar{F}_n\left(xU\left(\frac{n}{k}\right)\right)\frac{f'\left(xU\left(\frac{n}{k}\right)\right)f\left(xU\left(\frac{n}{k}\right)\right)}{f'\left(U\left(\frac{n}{k}\right)\right)f\left(U\left(\frac{n}{k}\right)\right)}\mathrm{d}x=1-\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}+o_P\left(\frac{1}{\sqrt{k}}\right)$$

$$\int_{1}^{\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}}\frac{n}{k}\bar{F}_n\left(xU\left(\frac{n}{k}\right)\right)\frac{f'\left(xU\left(\frac{n}{k}\right)\right)}{f'\left(U\left(\frac{n}{k}\right)\right)}\mathrm{d}x=\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}-1+o_P\left(\frac{1}{\sqrt{k}}\right)$$

再结合引理1和文献[7]的引理1，当 n 充分大时，可得

$$\frac{1}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}\left(\frac{1}{\bar{F}_n(X_{n-k,n})}\int_{X_{n-k,n}}^{U\left(\frac{n}{k}\right)}\bar{F}_n(v)f'(v)f(v)\mathrm{d}v+\frac{f\left(U\left(\frac{n}{k}\right)\right)}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\int_{U\left(\frac{n}{k}\right)}^{X_{n-k,n}}\bar{F}_n(v)f'(v)\mathrm{d}v\right)=o_P\left(\frac{1}{\sqrt{k}}\right)$$

接下来考虑等式(8)右边的第二项. 注意到

$$\left(\frac{1}{\bar{F}_n(X_{n-k,n})}-\frac{1}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\right)\frac{\int_{U\left(\frac{n}{k}\right)}^{\infty}\bar{F}_n(v)f'(v)f(v)\mathrm{d}v}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}=$$

$$\left(\frac{1}{\bar{F}_n(X_{n-k,n})}-\frac{1}{\bar{F}_n\left(U\left(\frac{n}{k}\right)\right)}\right)\cdot$$

$$\left(\frac{\frac{1}{2n}\sum_{i=1}^{n}\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}+\right.$$

$$\left.\frac{\frac{1}{n}\sum_{i=1}^{n}\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}\cdot f\left(U\left(\frac{n}{k}\right)\right)}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}\right)$$

再根据引理2，当 n 充分大时，可得

$$\frac{1}{\bar{F}\left(U\left(\frac{n}{k}\right)\right)}\frac{\frac{1}{2n}\sum_{i=1}^{n}\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}=$$

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2 \mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}{E\left(\left(f(X)-f\left(U\left(\frac{n}{k}\right)\right)\right)^2 \mathbb{1}_{\left\{X>U\left(\frac{n}{k}\right)\right\}}\right)}\to 1$$

从而有

$$\left(\frac{1}{\overline{F}_n(X_{n-k,n})}-\frac{1}{\overline{F}_n\left(U\left(\frac{n}{k}\right)\right)}\right)\frac{\int_{U\left(\frac{n}{k}\right)}^{\infty}\overline{F}_n(v)f'(v)f(v)\mathrm{d}v}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}=$$

$$\left(1-\frac{\overline{F}\left(U\left(\frac{n}{k}\right)\right)}{\overline{F}_n\left(U\left(\frac{n}{k}\right)\right)}\right)\left((1+o_P(1))+\frac{\frac{1}{n}\sum_{i=1}^{n}\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}\cdot f\left(U\left(\frac{n}{k}\right)\right)}{\overline{F}\left(U\left(\frac{n}{k}\right)\right)\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}\right)$$

最后考虑等式(8)右边的第四项. 根据$\sqrt{k}\left(\frac{X_{n-k,n}}{U\left(\frac{n}{k}\right)}-1\right)\xrightarrow{d}N(0,1)$，可得

$$\left(\frac{f\left(U\left(\frac{n}{k}\right)\right)}{\overline{F}_n\left(U\left(\frac{n}{k}\right)\right)}-\frac{f(X_{n-k,n})}{\overline{F}_n(X_{n-k,n})}\right)\frac{\int_{X_{n-k,n}}^{\infty}\overline{F}_n(v)f'(v)\mathrm{d}v}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}=$$

$$\left(\frac{1}{\overline{F}_n\left(U\left(\frac{n}{k}\right)\right)}-\frac{f(X_{n-k,n})}{f\left(U\left(\frac{n}{k}\right)\right)}\cdot\frac{1}{\overline{F}_n(X_{n-k,n})}\right)\frac{f\left(U\left(\frac{n}{k}\right)\right)\frac{1}{n}\sum_{i=1}^{n}(f(X_i)-f(X_{n-k,n}))\mathbb{1}_{\{X_i>X_{n-k,n}\}}}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}=$$

$$\left(\frac{\overline{F}\left(U\left(\frac{n}{k}\right)\right)}{\overline{F}_n\left(U\left(\frac{n}{k}\right)\right)}-1\right)\frac{f\left(U\left(\frac{n}{k}\right)\right)\frac{1}{n}\sum_{i=1}^{n}\left(f(X_i)-f\left(U\left(\frac{n}{k}\right)\right)\right)\mathbb{1}_{\left\{X_i>U\left(\frac{n}{k}\right)\right\}}}{\overline{F}\left(U\left(\frac{n}{k}\right)\right)\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}$$

综上所述，当 n 充分大时，可得

$$\frac{(\hat{AE}_f(X_{n-k,n}))^2}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}-1=\left(\frac{\left(\hat{AE}_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}{\left(AE_f\left(U\left(\frac{n}{k}\right)\right)\right)^2}-1\right)-\left(\frac{\overline{F}_n\left(U\left(\frac{n}{k}\right)\right)}{\overline{F}\left(U\left(\frac{n}{k}\right)\right)}-1\right)+o_P\left(\frac{1}{\sqrt{k}}\right)$$

定理 1 得证.

推论 1 的证明　令 $f(x)=\log x$ 时有$\hat{AE}_{\log}(X_{n-k,n})=\hat{\gamma}_n(k)$ 和$\hat{AE}_{\log}\left(U\left(\frac{n}{k}\right)\right)=\tilde{\gamma}_n(k)$. 再结合定理 1，可得

$$\hat{\gamma}_n(k)=\tilde{\gamma}_n(k)+o_P\left(\frac{1}{\sqrt{k}}\right)$$

推论 1 得证.

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