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相依时序逐日盯市条件尾期望的经验估计[①]

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摘要：Chen 等提出了逐日盯市在险价值，并在严平稳 ρ-混合相依情形下，证明了其经验估计量的大样本性质. 基于逐日盯市在险价值，定义了逐日盯市条件尾期望，并在时序满足严平稳强混合条件时，分别证明了经验估计量的强相合性及渐近正态性.

关 键 词：逐日盯市条件尾期望；强相合性；渐近正态性；经验估计；强混合

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Empirical Estimation on Mark to Market Conditional Tail Expectation under α-mixing Dependence Structure

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Abstract: Chen et al. (2018) proposed the mark to market value at risk, and proved the large sample properties of its empirical estimator when the time series satisfies the strictly stationary ρ-mixing condition. Based on the mark to market value at risk, this paper defines the mark to market conditional tail expectation, and proves the strong consistency as well as the asymptotic normality of their empirical estimators respectively on the premise of strictly stationary α-mixing sequence.

Key words: mark to market conditional tail expectation; strong consistency; asymptotic normality; empirical estimation; α-mixing

近年来，风险度量在金融市场中变得愈加重要. 记投资组合收益为随机变量 X，其累积分布函数为 $F_X(x)$. 给定显著性水平 $\alpha \in (0,1)$，在险价值

$$VaR_X(\alpha) = -\sup\{x: F_X(x) \leqslant \alpha\}$$

给出了投资组合以 α 的概率所遭受的最小损失值. 文献[1-2]分别刻画、研究了对冲基金的风险特征与资本充足率，及住宅市场的下行风险. 文献[3]将在险价值与医院金融风险管理相结合，以达增强医院资产流动的目的. 但 VaR 因不满足次可加性，故不是一致性风险度量[4]. 作为本文的研究对象，条件尾期望

$$CTE_X(\alpha) = -\mathrm{E}(X \mid X \leqslant -VaR_X(\alpha))$$

刻画了投资组合在损失超过阈值 $VaR_X(\alpha)$ 时所遭受的期望损失值，且当 $F_X(x)$ 为连续函数时，其满足一

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致性风险度量的所有理想性质[4-5]. 关于条件尾期望估计量及条件尾期望在银行、金融服务与保险等领域的广泛应用，见文献[6-12].

尽管在险价值与条件尾期望有着简单且易于理解的表达式，但其是根据持有期结束时的投资组合收益变化计算的. 故当金融机构出现投资组合在其持有期内有多次结算时，在险价值与条件尾期望很难适用于该金融机构内部风险的评估. 同时为缓解在险价值对资产市场风险的低估，基于投资组合在每一个连续交易日的收益时序$\{X_1, X_2, \cdots\}$，文献[13]提出了逐日盯市在险价值

$$MMVaR_{S_k^X}(\alpha) = -\sup\left\{x: \mathrm{P}\left(\bigcup_{i=1}^{k}\left\{\sum_{j=1}^{t} X_j > c,\ l=1,\cdots,i-1,\ \sum_{j=1}^{i} X_j < c\right\}\right) \leqslant \alpha\right\} = -\sup\{y: F_Y(y) \leqslant \alpha\}$$

其中：$S_k^X = X_1 + \cdots + X_k$，$k$ 为任一给定的正整数；$F_Y(y)$ 为当 $i=1$ 时，$Y_i := \min(X_i, X_i + X_{i+1}, \cdots, X_i + X_{i+1} + \cdots + X_{i+k-1})$ 的累积分布函数. 文献[13]在序列$\{X_i, i \geqslant 1\}$满足严平稳 ρ-混合相依性的条件下，证明了逐日盯市在险价值经验估计计量 $-\hat{F}_{Y,n}^{-1}(\alpha)$ 的强相合性与渐近正态性，其中$\hat{F}_{Y,n}^{-1}(\alpha)$ 为

$$\hat{F}_{Y,n}(y) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \leqslant y)$$

的 α 分位数. 基于逐日盯市在险价值，本文提出逐日盯市条件尾期望：

$$MMCTE_{S_k^X}(\alpha) = -\mathrm{E}(Y_1 \mid Y_1 \leqslant -MMVaR_{S_k^X}(\alpha))$$

为保证逐日盯市条件尾期望满足风险度量一致性公理，且为便于后续理论推导，本文针对 Y_1 的累积分布函数 F_Y，作如下假设：

(i) F_Y 在定义域内连续；

(ii) F_Y 在 ξ_α 的某邻域内至少二阶可导，其中 $\xi_\alpha := F_Y^{-1}(\alpha) = \inf\{y: F_Y(y) \geqslant \alpha\}$，且 F_Y 的一阶导函数 F'_Y与二阶导函数 F''_Y在该邻域内有界；

(iii) $F'_Y(\xi_\alpha) =: f(\xi_\alpha) > 0$.

易知上述假设(i) 保证了 $MMVaR_{S_k^X}(\alpha) = -\xi_\alpha$，且由假设(ii),(iii) 可知逐日盯市条件尾期望有如下积分表达式

$$MMCTE_{S_k^X}(\alpha) = -\frac{1}{\alpha}\int_{-\infty}^{\xi_\alpha} y\,\mathrm{d}F_Y(y) = -\frac{1}{\alpha}\int_0^{\alpha} F_Y^{-1}(u)\,\mathrm{d}u$$

其经验估计量定义为

$$MMCTE_{S_k^X,n}(\alpha) = -\frac{1}{\alpha}\int_0^{\alpha} \hat{F}_{Y,n}^{-1}(u)\,\mathrm{d}u = -\frac{1}{\alpha}\left(\frac{1}{n}\sum_{i=1}^{[n\alpha]} Y_{i:n} + \left(\frac{n\alpha - [n\alpha]}{n}\right) Y_{[n\alpha]+1:n}\right)$$

其中 $Y_{i:n}$ 为$Y_1, \cdots, Y_n$ 的第i 次序统计量，$[n\alpha]$表示对 $n\alpha$ 向下取整. 本文将在时序满足相较于文献[13]更弱的相依性，即严平稳强混合的条件下，证明 $-\hat{F}_{Y,n}^{-1}(\alpha)$ 与 $MMCTE_{S_k^X,n}(\alpha)$ 的强相合性与渐近正态性.

本文主要结论如下：

定理 1　假设$\{X_i, i \geqslant 1\}$ 为严平稳强混合序列，其混合系数满足 $\alpha(n) \leqslant cn^{-\beta}$，其中 $c>0$，$\beta>1$. 则对任一给定的 $\alpha \in (0,1)$ 与任意的 $\delta \in \left(\frac{9}{10+8\beta}, \frac{1}{2}\right)$，当 $n \to \infty$ 时，

$$\hat{F}_{Y,n}^{-1}(\alpha) = -MMVaR_{S_k^X}(\alpha) + O(n^{-\frac{1}{2}+\delta}) \tag{1}$$

几乎处处成立.

注 1　对比定理 1 与文献[14]中的引理 3.3 可知，基于相同的参数 δ，本文得到了一个比 $O(n^{-\frac{1}{2}+\delta}(\log\log n)^{\frac{1}{2}})$ 更好的界 $O(n^{-\frac{1}{2}+\delta})$.

定理 1 的证明　为书写方便，记$\hat{F}_{Y,n}^{-1}(\alpha)$ 为 $\hat{\xi}_\alpha$. 因$\{X_i, i \geqslant 1\}$ 为严平稳强混合序列，故由定义可知，

$\{Y_i, i \geqslant 1\}$ 也是严平稳强混合序列，且其强混合系数 $\alpha_Y(n)$ 满足 $\alpha_Y(n) \leqslant \alpha(n) \leqslant cn^{-\beta}$，其中 $c > 0$，$\beta > 1$. 对于任意的 $\delta \in \left(\frac{9}{10+\beta}, \frac{1}{2}\right)$，令 $\epsilon_n = n^{-\frac{1}{2}+\delta}$，$n \geqslant 1$，有

$$
\begin{aligned}
\mathrm{P}(\hat{\xi}_\alpha > \xi_\alpha + \epsilon_n) = \mathrm{P}(\alpha > \hat{F}_{Y,n}(\xi_\alpha + \epsilon_n)) = \mathrm{P}(1 - \hat{F}_{Y,n}(\xi_\alpha + \epsilon_n) > 1 - \alpha) = \\
\mathrm{P}(\sum_{i=1}^{n} (Y_i > \xi_\alpha + \epsilon_n) > n(1-\alpha)) = \\
\mathrm{P}(\sum_{i=1}^{n} (V_i - \mathrm{E}V_i) > n\upsilon_{n1})
\end{aligned}
$$

其中 $V_i = \mathrm{I}(Y_i > \xi_\alpha + \epsilon_n)$，$\upsilon_{n1} = F_Y(\xi_\alpha + \epsilon_n) - \alpha = f(\xi_\alpha) n^{-\frac{1}{2}+\delta} + o(\epsilon_n)$. 因为 $\{Y_i, i \geqslant 1\}$ 是严平稳强混合序列，由定义可知，$\{V_i - \mathrm{E}V_i, i \geqslant 1\}$ 亦为严平稳强混合序列，且与 $\{Y_i, i \geqslant 1\}$ 有着相同的强混合系数. 同时因其满足 $|V_i - \mathrm{E}V_i| \leqslant 1$，故由文献[15]的定理 1.3 可得

$$
\begin{aligned}
\mathrm{P}(\hat{\xi}_\alpha > \xi_\alpha + \epsilon_n) \leqslant \mathrm{P}\left(\left|\sum_{i=1}^{n} (V_i - \mathrm{E}V_i)\right| > n\upsilon_{n1}\right) \leqslant \\
4\exp\left(-\frac{q\upsilon_{n1}^2}{8}\right) + 22\left(1 + \frac{4}{\upsilon_{n1}}\right)^{\frac{1}{2}} q\alpha_Y\left(\left[\frac{n}{2q}\right]\right)
\end{aligned} \tag{2}
$$

类似地，有

$$
\mathrm{P}(\hat{\xi}_\alpha < \xi_\alpha - \epsilon_n) \leqslant 4\exp\left(-\frac{q\upsilon_{n2}^2}{8}\right) + 22\left(1 + \frac{4}{\upsilon_{n2}}\right)^{\frac{1}{2}} q\alpha_Y\left(\left[\frac{n}{2q}\right]\right) \tag{3}
$$

其中 $\upsilon_{n2} = \alpha - F_Y(\xi_\alpha - \epsilon_n) = f(\xi_\alpha) n^{-\frac{1}{2}+\delta} + o(\epsilon_n)$，$\left[\frac{n}{2q}\right]$ 表示对 $\frac{n}{2q}$ 向下取整. 记 $\upsilon_n = \min(\upsilon_{n1}, \upsilon_{n2})$，则当 n 充分大时，有不等式 $\upsilon_n \geqslant \frac{1}{3} f(\xi_\alpha) n^{-\frac{1}{2}+\delta}$ 成立. 令 $q = \lceil n^{1-2\delta} (\log n)^2 \rceil$，其中 $\lceil\ \rceil$ 表示向上取整，综合不等式(2)与(3)可知，对于充分大的 n，有

$$
\begin{aligned}
\mathrm{P}(|\hat{\xi}_\alpha - \xi_\alpha| > \epsilon_n) \leqslant 8\exp\left(-\frac{q\upsilon_n^2}{8}\right) + 44\left(1 + \frac{4}{\upsilon_n}\right)^{\frac{1}{2}} q\alpha_Y\left(\left[\frac{n}{2q}\right]\right) \leqslant \\
C_1 \exp(-(C_2 \log n) \log n) + C_3 \left(\frac{f(\xi_\alpha) n^{-\frac{1}{2}+\delta} + 12}{f(\xi_\alpha) n^{-\frac{1}{2}+\delta}}\right)^{\frac{1}{2}} \times \\
n^{1-2\delta} (\log n)^2 \left(\frac{n^{1-2\delta} (\log n)^2}{n}\right)^{\beta} \leqslant \\
\frac{C_1}{n^2} + C_2 \frac{(\log n)^{2(1+\beta)}}{n^{\tau}}
\end{aligned}
$$

成立，其中 $\tau = -\frac{5}{4} + \delta\left(\frac{5}{2} + 2\beta\right) > 1$. 因此，存在正整数 n_0 使得

$$
\sum_{n=1}^{\infty} \mathrm{P}(|\hat{\xi}_\alpha - \xi_\alpha| > \epsilon_n) \leqslant C_3 + C_1 \sum_{n=n_0}^{\infty} \frac{1}{n^2} + C_2 \sum_{n=n_0}^{\infty} \frac{(\log n)^{2(1+\beta)}}{n^{\tau}} < \infty
$$

则由 Borel-Cantelli 引理可知(1)式几乎处处成立. 定理证毕.

如下定理 2 给出了逐日盯市在险价值经验估计量的渐近分布.

定理 2 假设 $\{X_i, i \geqslant 1\}$ 为严平稳强混合序列，其混合系数满足 $\alpha(n) \leqslant cn^{-\beta}$，其中 $c > 0$，$\beta > 3$. 记 $Y_{(1)} \leqslant Y_{(2)} \leqslant \cdots \leqslant Y_{(n)} \leqslant \cdots$ 为过程 $\{Y_i, i \geqslant 1\}$ 的次序统计量. 若正整数序列 $\{k_n, n \geqslant 1\}$ 满足 $1 \leqslant k_n \leqslant n$，且当 $n \to \infty$ 时，

$$
k_n = n\alpha + o(n^{\frac{1}{2}})
$$

记 $H_n = Y_{(k_n)}$，则

$$
\frac{n^{\frac{1}{2}} f(\xi_\alpha)(H_n - \xi_\alpha)}{\sigma_\alpha} \xrightarrow{\mathscr{L}} N(0, 1) \tag{4}
$$

其中 $\sigma_{\alpha}^{2}=\lim\limits_{n\to\infty} n\operatorname{Var}(\hat{F}_{Y,n}(\xi_{\alpha}))<\infty$.

证　因为

$$\mathrm{P}(H_n\geqslant\xi_\alpha+a_n)=\mathrm{P}\Big(\sum_{i=1}^{n}\mathrm{I}(Y_i\leqslant\xi_\alpha+a_n)\leqslant k_n\Big)=\mathrm{P}\Big(\sum_{i=1}^{n}\mathrm{I}(Y_i>\xi_\alpha+a_n)\geqslant n-k_n\Big)=$$
$$\mathrm{P}\Big(\sum_{i=1}^{n}(V_i-\mathrm{E}V_i)\geqslant n\tilde{v}_{n1}\Big)$$

其中 $V_i=\mathrm{I}(Y_i>\xi_\alpha+a_n)$，$\tilde{v}_{n1}=F_Y(\xi_\alpha+a_n)-\dfrac{k_n}{n}=F_Y(\xi_\alpha+a_n)-\alpha+o(n^{-\frac{1}{2}})=f(\xi_\alpha)a_n+o(a_n)$，及

$$\mathrm{P}(H_n\leqslant\xi_\alpha-a_n)=\mathrm{P}\Big(\sum_{i=1}^{n}(W_i-\mathrm{E}W_i)\geqslant n\tilde{v}_{n2}\Big)$$

其中：$W_i=\mathrm{I}(Y_i\leqslant\xi_\alpha-a_n)$，$\tilde{v}_{n2}=\dfrac{k_n}{n}-F_Y(\xi_\alpha-a_n)=f(\xi_\alpha)a_n+o(a_n)$. 则由文献[16]引理 3.4 的证明易得，当 n 充分大时，$H_n\in\mathscr{D}_n:=[\xi_\alpha-a_n,\ \xi_\alpha+a_n]$ 几乎处处成立，其中 $a_n=n^{-\frac{1}{2}}(\log\log n\cdot\log n)^{\frac{1}{2}}$. 由文献[16]的定理 2.4 可知，当 $\delta>\max\left(\dfrac{5}{\beta-3}-1,\ \dfrac{2}{\beta-1}\right)$，$\beta>3$，及 $n\to\infty$ 时，对所有的 $y\in\mathscr{D}_n$，有

$$\hat{F}_{Y,n}(y)-\hat{F}_{Y,n}(\xi_\alpha)-(F_Y(y)-\alpha)=O\left(n^{-\frac{3}{4}+\frac{\delta}{4(2+\delta)}}(\log\log n\cdot\log n)^{\frac{1}{2}}\right)\tag{5}$$

几乎处处成立. 因此，基于(5)式可得，当 $n\to\infty$ 时，

$$\frac{n^{\frac{1}{2}}(\hat{F}_{Y,n}(H_n)-F_Y(H_n))}{\sigma_\alpha}=\frac{n^{\frac{1}{2}}(\hat{F}_{Y,n}(\xi_\alpha)-\alpha)}{\sigma_\alpha}+o(1)\tag{6}$$

几乎处处成立，其中由文献[15]的定理 1.5 与定理 1.7 可知 $\dfrac{n^{\frac{1}{2}}(\hat{F}_{Y,n}(\xi_\alpha)-\alpha)}{\sigma_\alpha}\xrightarrow{\mathscr{L}}N(0,1)$，$\sigma_\alpha^2=\lim\limits_{n\to\infty} n\operatorname{Var}(\hat{F}_{Y,n}(\xi_\alpha))<\infty$. 利用泰勒展式，有

$$\begin{aligned}\frac{n^{\frac{1}{2}}(\hat{F}_{Y,n}(H_n)-F_Y(H_n))}{\sigma_\alpha}&=\frac{n^{\frac{1}{2}}(F_Y(\xi_\alpha)-F_Y(H_n)+o(n^{-\frac{1}{2}}))}{\sigma_\alpha}=\\&\frac{n^{\frac{1}{2}}f(\xi_\alpha)(\xi_\alpha-H_n)}{\sigma_\alpha}+\frac{n^{\frac{1}{2}}O(a_n^2)}{2\sigma_\alpha}+o(1)=\\&\frac{n^{\frac{1}{2}}f(\xi_\alpha)(\xi_\alpha-H_n)}{\sigma_\alpha}+o(1)\end{aligned}\tag{7}$$

综合(6),(7)式可知(4)式成立. 定理证毕.

接下来的定理 3－4 给出了 $MMCTE_{S_k^X}(\alpha)$ 经验估计量的强相合性与渐近正态性.

定理 3　假设定理 1 的条件成立，且 Y_1 的一阶矩存在. 则对任一固定的 $\alpha\in(0,1)$，$MMCTE_{S_k^X,n}(\alpha)\xrightarrow{a.s.}MMCTE_{S_k^X}(\alpha)$.

证　若证得

$$\int_0^1|\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u)|\,\mathrm{d}u\xrightarrow{a.s.}0\tag{8}$$

易知该定理结论成立. 因为 $\mathrm{E}|Y_1|<\infty$，即 $\int_0^1|F_Y^{-1}(u)|\,\mathrm{d}u<\infty$，则对任意给定的 $\epsilon>0$，存在充分小的 $\delta_\epsilon>0$，

$$\int_{[\delta_\epsilon,\ 1-\delta_\epsilon]^C}|F_Y^{-1}(u)|\,\mathrm{d}u<\frac{\epsilon}{5}\tag{9}$$

(8)式的左边部分可拆分为

$$\int_0^1|\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u)|\,\mathrm{d}u=\int_{[\delta_\epsilon,\ 1-\delta_\epsilon]^C}|\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u)|\,\mathrm{d}u+\int_{\delta_\epsilon}^{1-\delta_\epsilon}|\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u)|\,\mathrm{d}u\leqslant$$

$$\int_{[\delta_\epsilon,\ 1-\delta_\epsilon]^C} |\hat{F}_{Y,n}^{-1}(u)|\mathrm{d}u + \int_{[\delta_\epsilon,\ 1-\delta_\epsilon]^C} |F_Y^{-1}(u)|\mathrm{d}u +$$
$$\int_{\delta_\epsilon}^{1-\delta_\epsilon} |\hat{F}_{Y,n}^{-1}(u) - F_Y^{-1}(u)|\mathrm{d}u =$$
$$\int_0^1 (|\hat{F}_{Y,n}^{-1}(u)| - |F_Y^{-1}(u)|)\mathrm{d}u - \int_{\delta_\epsilon}^{1-\delta_\epsilon} (|\hat{F}_{Y,n}^{-1}(u)| - |F_Y^{-1}(u)|)\mathrm{d}u +$$
$$2\int_{[\delta_\epsilon,\ 1-\delta_\epsilon]^C} |F_Y^{-1}(u)|\mathrm{d}u + \int_{\delta_\epsilon}^{1-\delta_\epsilon} |\hat{F}_{Y,n}^{-1}(u) - F_Y^{-1}(u)|\mathrm{d}u =:$$
$$I_n^{(1)} + I_n^{(2)} + I^{(3)} + I_n^{(4)} \tag{10}$$

其中由定理 1 的证明易得 $I_n^{(1)} = \dfrac{1}{n}\sum_{i=1}^{n} |Y_i| - \mathrm{E}|Y_1| \xrightarrow{a.s.} 0$，因此存在 $N_1 > 0$，使得对于所有的 $n > N_1$，有

$$\mathrm{P}(|I_n^{(1)}| < \frac{\epsilon}{5}) = 1 \tag{11}$$

由定理 1 可知，$\hat{F}_{Y,n}^{-1}(u) \xrightarrow{a.s.} F_Y^{-1}(u)$ 在区间 $u \in (0,1)$ 上局部一致成立. 故利用控制收敛定理，对于上述给定的 ϵ，存在 $N_2 > 0$，使得对于所有的 $n > N_2$，有

$$\mathrm{P}\left(|I_n^{(2)}| < \frac{\epsilon}{5}\right) = 1 \tag{12}$$

与

$$\mathrm{P}\left(|I_n^{(4)}| < \frac{\epsilon}{5}\right) = 1 \tag{13}$$

成立. 综合(9)−(13)式可知对任意给定的 $\epsilon > 0$，存在 $N_{\epsilon,\delta} := \max(N_1, N_2) > 0$，使得对所有的 $n > N_{\epsilon,\delta}$，有 $\mathrm{P}\left(\left|\int_0^1 |\hat{F}_{Y,n}^{-1}(u) - F_Y^{-1}(u)|\mathrm{d}u\right| < \epsilon\right) = 1$. 这表明(8)式成立. 定理证毕.

定理 4 假设定理 1 的条件成立，则对任一给定的 $\alpha \in (0,1)$，

$$n^{\frac{1}{2}}(MMCTE_{S_k^X,\,n}(\alpha) - MMCTE_{S_k^X}(\alpha)) \xrightarrow{\mathscr{L}} N(0, \sigma_Y^2(\alpha))$$

其中 $\sigma_Y^2(\alpha) = \dfrac{1}{\alpha^2}\lim\limits_{n\to\infty}\mathrm{Var}\left(\dfrac{1}{n^{\frac{1}{2}}}\sum_{i=1}^{n}\int_{-\infty}^{\xi_\alpha}(\mathrm{I}(Y_i \leqslant u) - F_Y(u))\mathrm{d}u\right) < \infty$.

证 首先计算如下积分：

$$\int_0^\alpha \hat{F}_{Y,n}^{-1}(u)\mathrm{d}u = \int_0^\alpha \hat{F}_{Y,n}^{-1}(u)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0)\mathrm{d}u + \int_0^\alpha \hat{F}_{Y,n}^{-1}(u)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}u =$$
$$-\int_0^\alpha\int_{\hat{F}_{Y,n}^{-1}(u)}^0 \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0)\mathrm{d}s\mathrm{d}u + \int_{(0,\ \hat{F}_{Y,n}^{-1}(0)]} \hat{F}_{Y,n}^{-1}(u)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}u +$$
$$\int_{(\hat{F}_{Y,n}^{-1}(0),\ \alpha]} \hat{F}_{Y,n}^{-1}(u)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}u =$$
$$-\int_{-\infty}^{\hat{F}_{Y,n}^{-1}(\alpha)}\int_0^{\hat{F}_{Y,n}^{-1}(s)} \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0)\mathrm{d}u\mathrm{d}s - \int_{\hat{F}_{Y,n}^{-1}(\alpha)}^0\int_0^\alpha \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0)\mathrm{d}u\mathrm{d}s -$$
$$\int_{(0,\ \hat{F}_{Y,n}^{-1}(0)]}\int_{\hat{F}_{Y,n}^{-1}(u)}^0 \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}s\mathrm{d}u + \int_{(\hat{F}_{Y,n}^{-1}(0),\ \alpha]}\int_0^{\hat{F}_{Y,n}^{-1}(u)} \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}s\mathrm{d}u =$$
$$-\int_{-\infty}^{\hat{F}_{Y,n}^{-1}(\alpha)} \hat{F}_{Y,n}(s)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0)\mathrm{d}s + \alpha\hat{F}_{Y,n}^{-1}(\alpha)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0) -$$
$$\int_{-\infty}^0\int_0^{\hat{F}_{Y,n}^{-1}(s)} \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}u\mathrm{d}s + \int_0^{\hat{F}_{Y,n}^{-1}(\alpha)}\int_{\hat{F}_{Y,n}^{-1}(s)}^\alpha \mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}u\mathrm{d}s =$$
$$-\int_{-\infty}^{\hat{F}_{Y,n}^{-1}(\alpha)} \hat{F}_{Y,n}(s)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0)\mathrm{d}s + \alpha\hat{F}_{Y,n}^{-1}(\alpha)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) < 0) - \int_{-\infty}^0 \hat{F}_{Y,n}(s) \times$$
$$\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}s - \int_0^{\hat{F}_{Y,n}^{-1}(\alpha)} \hat{F}_{Y,n}(s)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0)\mathrm{d}s + \alpha\hat{F}_{Y,n}^{-1}(\alpha)\mathrm{I}(\hat{F}_{Y,n}^{-1}(\alpha) \geqslant 0) =$$

$$\alpha\hat{F}_{Y,n}^{-1}(\alpha)-\int_{-\infty}^{\hat{F}_{Y,n}^{-1}(\alpha)}\hat{F}_{Y,n}(s)\mathrm{d}s \tag{14}$$

基于类似的计算步骤，可得

$$\int_0^\alpha F_Y^{-1}(u)\mathrm{d}u=\alpha F_Y^{-1}(\alpha)-\int_{-\infty}^{F_Y^{-1}(\alpha)}F_Y(s)\mathrm{d}s \tag{15}$$

接着计算如下等式：

$$\begin{aligned}
&MMCTE_{S_k^X,n}(\alpha)-MMCTE_{S_k^X}(\alpha)=\\
&-\frac{1}{\alpha}\int_0^\alpha(\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u))\mathrm{d}u=\\
&-\frac{1}{\alpha}\left(\int_0^1(\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u))\mathrm{d}u-\int_\alpha^1(\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u))\mathrm{d}u\right)=\\
&-\frac{1}{\alpha}\left(-\int_{-\infty}^{\infty}(\hat{F}_{Y,n}(u)-F_Y(u))\mathrm{d}u-\int_\alpha^1(\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u))\mathrm{d}u\right)=\\
&-\frac{1}{\alpha}\left(-\int_{-\infty}^{F_Y^{-1}(\alpha)}(\hat{F}_{Y,n}(u)-F_Y(u))\mathrm{d}u+R_{Y,n}(\alpha)\right)
\end{aligned}$$

其中基于(14) 与(15) 式可得

$$\begin{aligned}
R_{Y,n}(\alpha)=&-\int_{F_Y^{-1}(\alpha)}^{\infty}(\hat{F}_{Y,n}(u)-F_Y(u))\mathrm{d}u-\int_\alpha^1(\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u))\mathrm{d}u=\\
&\int_0^\alpha(\hat{F}_{Y,n}^{-1}(u)-F_Y^{-1}(u))\mathrm{d}u+\int_{-\infty}^{F_Y^{-1}(\alpha)}(\hat{F}_{Y,n}(u)-F_Y(u))\mathrm{d}u=\\
&\int_0^\alpha\hat{F}_{Y,n}^{-1}(u)\mathrm{d}u+\int_{-\infty}^{F_Y^{-1}(\alpha)}\hat{F}_{Y,n}(u)\mathrm{d}u-\alpha F_Y^{-1}(\alpha)=\\
&-\int_{-\infty}^{\hat{F}_{Y,n}^{-1}(\alpha)}\hat{F}_{Y,n}(y)\mathrm{d}y+\alpha\hat{F}_{Y,n}^{-1}(\alpha)+\int_{-\infty}^{F_Y^{-1}(\alpha)}\hat{F}_{Y,n}(u)\mathrm{d}u-\alpha F_Y^{-1}(\alpha)=\\
&\int_{\hat{F}_{Y,n}^{-1}(\alpha)}^{F_Y^{-1}(\alpha)}(\hat{F}_{Y,n}(y)-\alpha)\mathrm{d}y
\end{aligned}$$

由此可知 $R_{Y,n}(\alpha)$ 的界为

$$|R_{Y,n}(\alpha)|\leqslant|\hat{F}_{Y,n}(F_Y^{-1}(\alpha))-\alpha||F_Y^{-1}(\alpha)-\hat{F}_{Y,n}^{-1}(\alpha)|+|\hat{F}_{Y,n}(\hat{F}_{Y,n}^{-1}(\alpha))-\alpha||F_Y^{-1}(\alpha)-\hat{F}_{Y,n}^{-1}(\alpha)|$$

基于此界，对任一给定的 $\alpha\in(0,1)$，因为 $\hat{F}_{Y,n}(F_Y^{-1}(\alpha))-\alpha=O_P\left(\frac{1}{n^{\frac{1}{2}}}\right)$，$\hat{F}_{Y,n}(\hat{F}_{Y,n}^{-1}(\alpha))-\alpha=O\left(\frac{1}{n}\right)$，且由定理 1 可知 $F_Y^{-1}(\alpha)-\hat{F}_{Y,n}^{-1}(\alpha)=o_P(1)$，故 $n^{\frac{1}{2}}|R_{Y,n}(\alpha)|=o_P(1)$. 因此，对任一给定的 $\alpha\in(0,1)$，利用文献[15] 的定理 1.5 与定理 1.7，有

$$\begin{aligned}
&n^{\frac{1}{2}}(MMCTE_{S_k^X,n}(\alpha)-MMCTE_{S_k^X}(\alpha))=\\
&\frac{1}{\alpha}\int_{-\infty}^{F_Y^{-1}(\alpha)}n^{\frac{1}{2}}(\hat{F}_{Y,n}(u)-F_Y(u))\mathrm{d}u-\frac{1}{\alpha}(n^{\frac{1}{2}}R_{Y,n}(\alpha))=\\
&n^{\frac{1}{2}}\left(\frac{1}{\alpha}\int_{-\infty}^{F_Y^{-1}(\alpha)}\left(\frac{1}{n}\sum_{i=1}^n(\mathrm{I}(Y_i\leqslant u)-F_Y(u))\right)\mathrm{d}u\right)+o_P(1)=\\
&\frac{1}{n^{\frac{1}{2}}}\sum_{i=1}^n\left(\frac{1}{\alpha}\int_{-\infty}^{\xi_\alpha}(\mathrm{I}(Y_i\leqslant u)-F_Y(u))\mathrm{d}u\right)+o_P(1)\xrightarrow{\mathscr{L}}\\
&N(0,\sigma_Y^2(\alpha))
\end{aligned}$$

成立，其中 $\sigma_Y^2(\alpha)=\frac{1}{\alpha^2}\lim\limits_{n\to\infty}\mathrm{Var}\left(\frac{1}{n^{\frac{1}{2}}}\sum_{i=1}^n\int_{-\infty}^{\xi_\alpha}(\mathrm{I}(Y_i\leqslant u)-F_Y(u))\mathrm{d}u\right)<\infty$. 定理证毕.

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