DOI:10.13718/j.cnki.xsxb.2023.06.008

一类半参尾指数估计量的渐近性质[①]

余乐乐， 彭作祥

西南大学 数学与统计学院，重庆 400715

摘要： 基于对数函数和幂函数构造的统计量，本文提出一类半参尾指数估计量，并证明其相合性和渐近正态性.

关 键 词： 极值估计量；重尾；相合性；渐近正态性

中图分类号： O211.4 **文献标志码：** A **文章编号：** 1000-5471(2023)06-0054-05

Asymptotic Properties of a Class of Semi-parametric Tail Exponential Estimators

YU Lele， PENG Zuoxiang

School of Mathematics and Statistics of Southwest University，ChongQing 400715，China

Abstract: Based on the statistics constructed by logarithmic function and power function, this paper proposes a class of semi parametric tail exponential estimators, and proves their consistency and asymptotic normality.

Key words: extreme value estimator; heavy tail; consistency; asymptotic normality

设$\{X_n, n \geqslant 1\}$为独立同分布的随机变量序列，其分布函数为$F(x)$. $X_{1,n} \leqslant \cdots \leqslant X_{n,n}$表示$X_1,\cdots,X_n$的次序统计量. 若存在规范化常数$a_n > 0$和$b_n$及非退化分布函数$G_\gamma(x)$使得

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G_\gamma(x) \tag{1}$$

由文献[1-2]可知

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-\frac{1}{\gamma}}\},\ \gamma \in \mathbb{R},\ 1+\gamma x \geqslant 0$$

则称F属于G的吸引场，记为$F \in D(G)$，γ为极值指数. 令$U(t) = F^{\leftarrow}\left(1-\dfrac{1}{t}\right)$，$t \geqslant 1$. 当$\gamma > 0$时，(1)式等价于

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma,\ x > 0 \tag{2}$$

---

① 收稿日期：2022-09-05
作者简介：余乐乐，硕士研究生，主要从事极值统计分析研究.
通信作者：彭作祥，教授.

文献[3]提出了著名的 Hill 估计量. 文献[4]为减小 Hill 估计量的偏差，构造了矩率估计量. 文献[5]利用函数 $g_{r,u}(x)=x^r\ln^u(x)$，$x\geqslant 1$ 构造出如下的统计量

$$G_n(k,r,u)=\frac{1}{k}\sum_{i=0}^{k-1}g_{r,u}\left(\frac{X_{n-i,n}}{X_{n-k,n}}\right) \tag{3}$$

其中 $\gamma r<1$，$u>-1$. 利用(3)式可以将 Hill 估计量、矩率估计量表示出来：

$$\hat{\gamma}_n^H(k)=G_n(k,0,1)$$

$$\hat{\gamma}_n^{MR}(k)=\frac{G_n(k,0,2)}{2G_n(k,0,1)}$$

极值指数估计的应用非常广泛，相关研究可参见文献[6-10].

本文利用统计量 $G_n(k,r,u)$ 构造如下的尾指数估计量

$$\hat{\gamma}_n(k,r)=\frac{G_n(k,r,1)}{G_n^2(k,r,0)}$$

假定(2)式成立且存在序列 $k=k(n)$ 满足当 $n\to\infty$ 时，

$$k(n)\to\infty,\ \frac{n}{k(n)}\to\infty \tag{4}$$

考虑 $\hat{\gamma}_n(k,r)$ 的弱相合性. 此外，如果存在可测函数 $A(t)$ 使得

$$\lim_{t\to\infty}\frac{\frac{U(tx)}{U(t)}-x^\gamma}{A(t)}=x^\gamma\frac{x^\rho-1}{\rho} \tag{5}$$

成立，则我们讨论 $\hat{\gamma}_n(k,r)$ 的渐近分布，其中 $\rho<0$ 表示二阶参数.

1　相合性和渐近正态性

定理 1　假定 $\gamma r<1$ 成立，在(2)式和(4)式的条件下，$\hat{\gamma}_n(k,r)\xrightarrow{\mathrm{P}}\gamma$.

定理 2　假定 $\gamma r<\frac{1}{2}$ 成立，在(4)式和(5)式的条件下，存在 $\lambda\in\mathbb{R}$ 使得

$$\lim_{n\to\infty}\sqrt{k}A\left(\frac{n}{k}\right)=\lambda \tag{6}$$

则当 $n\to\infty$ 时，

$$\sqrt{k}(\hat{\gamma}_n(k,r)-\gamma)\xrightarrow{\mathrm{d}}N(\mu(r),\sigma^2(r))$$

其中

$$\mu(r)=\frac{\lambda(1-2\gamma r)(1-\gamma r-\rho)+\gamma r(1-\gamma r)}{(1-\gamma r-\rho)^2}$$

$$\sigma^2(r)=\frac{\gamma^2(4\gamma^2r^2(1-2\gamma r)-4\gamma r(1-\gamma r-\gamma^2r^2))}{(1-2\gamma r)^2}+\frac{\gamma^2(1-2\gamma r+2\gamma^4r^4)}{(1-2\gamma r)^3}$$

2　定理的证明

设 $\{Y_n,n\geqslant 1\}$ 为独立同分布的标准 Pareto 序列，$Y_{1,n},\cdots,Y_{n,n}$ 表示 $Y_1,\cdots,Y_n$ 的次序统计量，由文献[11]可得到 $\{X_i\}_{i=1}^n\overset{\mathrm{d}}{=}\{U(Y_i)\}_{i=1}^n$，$\left\{\frac{Y_{n-i,n}}{Y_{n-k,n}}\right\}_{i=0}^{k-1}\overset{\mathrm{d}}{=}\{Y_{k-i,k}\}_{i=0}^{k-1}$.

定理 1 的证明　由文献[5]的定理 1 可知，当 $n\to\infty$ 时，

$$G_n(k,r,u)\xrightarrow{\mathrm{P}}\frac{\gamma^u\gamma(1+u)}{(1-\gamma r)^{1+u}}$$

得到

$$G_n(k,r,1)\xrightarrow{\mathrm{P}}\frac{\gamma}{(1-\gamma r)^2},\ G_n(k,r,0)\xrightarrow{\mathrm{P}}\frac{1}{(1-\gamma r)}$$

利用连续映射定理[12] 和 Slutsky 定理[13]，定理得证.

对定理 2 的证明，我们需要下面的辅助引理.

引理 1 在定理 2 的条件下，当 $n\to\infty$ 时，有

$$\sqrt{k}\left\{G_n(k,r,0)-\frac{1}{1-\gamma r},\ G_n(k,r,1)-\frac{\gamma}{(1-\gamma r)^2}\right\}\xrightarrow{\mathrm{d}}(\lambda\mu_1(r)+N_1,\ \lambda\mu_2(r)+N_2) \tag{7}$$

其中

$$\mu_1(r)=\frac{r}{(1-\gamma r)(1-\gamma r-\rho)}$$

$$\mu_2(r)=\frac{1-\gamma^2r^2-\rho}{(1-\gamma r)^2(1-\gamma r-\rho)^2}$$

(N_1,N_2) 是二维零均值高斯向量，满足

$$(N_1,N_2)\sim N(0,\boldsymbol{\Sigma}),\ \boldsymbol{\Sigma}=\begin{pmatrix}s_1^2 & s_{12}\\ s_{12} & s_2^2\end{pmatrix}$$

其中

$$s_1^2=\frac{\gamma^2r^2}{(1-\gamma r)^2(1-2\gamma r)}$$

$$s_2^2=\frac{\gamma^2(1-2\gamma r+2\gamma^4r^4)}{(1-\gamma r)^4(1-2\gamma r)^3}$$

$$s_{12}=\frac{\gamma^2r(1-\gamma r-\gamma^2r^2)}{(1-\gamma r)^3(1-2\gamma r)^2}$$

证 由二阶正规变换条件(5)式知，对充分大的 t，

$$\left(\frac{U(tx)}{U(t)}\right)^r=x^{\gamma r}\left(1+rA(t)\frac{x^\rho-1}{\rho}+o(A(t))\right),\ \ln\left(\frac{U(tx)}{U(t)}\right)=\gamma\ln x+A(t)\frac{x^\rho-1}{\rho}+o(A(t))$$

则

$$G_n(k,r,0)\xrightarrow{\mathrm{d}}\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}+rA\left(\frac{n}{k}\right)\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}\frac{Y_i^\rho-1}{\rho}+o\left(A\left(\frac{n}{k}\right)\right) \tag{8}$$

$$\begin{aligned}G_n(k,r,1)\xrightarrow{\mathrm{d}}\ &\gamma\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}\ln Y_i+A\left(\frac{n}{k}\right)\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}\frac{Y_i^\rho-1}{\rho}+\\ &\gamma rA\left(\frac{n}{k}\right)\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}\frac{Y_i^\rho-1}{\rho}\ln Y_i+o\left(A\left(\frac{n}{k}\right)\right)\end{aligned} \tag{9}$$

利用文献[14] 中的 Cramer-Wold 定理证明(7) 式成立. 对任意 $(\varphi,\psi)\in\mathbb{R}^2$，有

$$\sqrt{k}\left\{\varphi\left(G_n(k,r,0)-\frac{1}{1-\gamma r}\right)+\psi\left(G_n(k,r,0)-\frac{\gamma}{(1-\gamma r)^2}\right)\right\}\xrightarrow{\mathrm{d}}\sqrt{k}B_n(k,r)+\sqrt{k}C_n(k,r)+o_p(1) \tag{10}$$

其中

$$B_n(k,r)=\frac{1}{k}\sum_{i=1}^{k}\left\{\varphi\left(Y_i^{\gamma r}-\frac{1}{1-\gamma r}\right)+\psi\gamma\left(Y_i^{\gamma r}\ln Y_i-\frac{1}{(1-\gamma r)^2}\right)\right\}$$

$$C_n(k,r)=\frac{1}{k}\sum_{i=1}^{k}\left\{\varphi rA\left(\frac{n}{k}\right)Y_i^{\gamma r}\frac{Y_i^\rho-1}{\rho}+\psi\left(\gamma rA\left(\frac{n}{k}\right)Y_i^{\gamma r}\frac{Y_i^\rho-1}{\rho}\ln Y_i+A\left(\frac{n}{k}\right)Y_i^{\gamma r}\frac{Y_i^\rho-1}{\rho}\right)\right\}$$

当 $\gamma r<\dfrac{1}{2}$ 时，

$$\mathrm{E}\left(\varphi\left(Y_i^{\gamma r}-\frac{1}{1-\gamma r}\right)+\psi\gamma\left(Y_i^{\gamma r}\ln Y_i-\frac{1}{(1-\gamma r)^2}\right)\right)=\varphi^2 s_1^2+2\varphi\psi s_{12}+\psi^2 s_2^2$$

由列为林德伯格中心极限定理可得

$$\sqrt{k}B_n(k,r)\xrightarrow{\mathrm{d}}\varphi N_1+\psi N_2 \tag{11}$$

与文献[15]引理1类似计算，有

$$\sqrt{k}C_n(k,r)\xrightarrow{\mathrm{P}}\lambda(\varphi\mu_1(r)+\psi\mu_2(r)) \tag{12}$$

由(11)式,(12)式及 Slutsky 定理，知

$$\sqrt{k}(B_n(k,r)+C_n(k,r))\xrightarrow{\mathrm{d}}\varphi(N_1+\lambda\mu_1(r))+\psi(N_2+\lambda\mu_2(r)) \tag{13}$$

结合(10)式和(13)式，引理得证.

定理 2 的证明　定义

$$W_n^{(1)}:=\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}-\frac{1}{1-\gamma r}\right)$$

$$W_n^{(2)}:=\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k}Y_i^{\gamma r}\ln Y_i-\frac{1}{(1-\gamma r)^2}\right)$$

利用泰勒展式，(8)式和(9)式化简为

$$\frac{1}{G_n^2(k,r,0)}\stackrel{\mathrm{d}}{=}(1-\gamma r)^2\left(1-2\left(\frac{1-\gamma r}{\sqrt{k}}W_n^{(1)}+\frac{rA\left(\frac{n}{k}\right)}{1-\gamma r-\rho}\right)\right)(1+o_p(1))$$

$$G_n(k,r,1)\stackrel{\mathrm{d}}{=}\frac{\gamma}{(1-\gamma r)^2}\left(1+\frac{(1-\gamma r)^2}{\sqrt{k}}W_n^{(2)}+\frac{(1-\gamma r)A\left(\frac{n}{k}\right)}{\gamma(1-\gamma r-\rho)}+\frac{r(2-2\gamma r-\rho)A\left(\frac{n}{k}\right)}{(1-\gamma r-\rho)^2}+o\left(A\left(\frac{n}{k}\right)\right)\right)$$

得到

$$\frac{G_n(k,r,1)}{G_n^2(k,r,0)}\stackrel{\mathrm{d}}{=}\gamma\left(1+\frac{(1-\gamma r)^2}{\sqrt{k}}W_n^{(2)}-\frac{2(1-\gamma r)}{\sqrt{k}}W_n^{(1)}+\frac{(1-2\gamma r)(1-\gamma r-\rho)+\gamma r(1-\gamma r)}{\gamma(1-\gamma r-\rho)^2}A\left(\frac{n}{k}\right)+o\left(A\left(\frac{n}{k}\right)\right)+o_p\left(\frac{1}{\sqrt{k}}\right)\right)$$

即

$$\sqrt{k}(\hat{\gamma}_n(k,r)-\gamma)\stackrel{\mathrm{d}}{=}\gamma(1-\gamma r)^2W_n^{(2)}-2\gamma(1-\gamma r)W_n^{(1)}+\frac{(1-2\gamma r)(1-\gamma r-\rho)+\gamma r(1-\gamma r)}{(1-\gamma r-\rho)^2}A\left(\frac{n}{k}\right)+o\left(A\left(\frac{n}{k}\right)\right)+o_p\left(\frac{1}{\sqrt{k}}\right)$$

由引理 1 知

$$(W_n^{(1)},\gamma W_n^{(2)})\xrightarrow{\mathrm{d}}(N_1,N_2)$$

结合(6)式，定理 2 得证.

参考文献:

[1] DE HAAN L, FERREIRA A. Extreme Value Theory [M]. New York: Springer, 2006.

[2] RESNICK S I. Point Processes [M]//Extreme Values, Regular Variation and Point Processes. New York: Springer, 1987: 123-161.

[3] HILL B M. A Simple General Approach to Inference about the Tail of a Distribution [J]. The Annals of Statistics, 1975, 3(5): 1163-1174.

[4] DANIELSSON J, JANSEN D W, DE VRIES C G. The Method of Moments Ratio Estimator for the Tail Shape Parameter [J]. Communications in Statistics-Theory and Methods, 1996, 25(4): 711-720.

[5] PAULAUSKAS V, VAIČIULIS M. A Class of New Tail Index Estimators [J]. Annals of the Institute of Statistical Mathematics, 2017, 69(2): 461-487.

[6] QI Y C. On the Tail Index of a Heavy Tailed Distribution [J]. Annals of the Institute of Statistical Mathematics, 2010, 62(2): 277-298.

[7] PAULAUSKAS V, VAIČIULIS M. Comparison of the Several Parameterized Estimators for the Positive Extreme Value Index [J]. Journal of Statistical Computation and Simulation, 2017, 87(7): 1342-1362.

[8] VAIČIULIS M, MARKOVICH N M. A Class of Semiparametric Tail Index Estimators and Its Applications [J]. Automation and Remote Control, 2019, 80(10): 1803-1816.

[9] BERAN J, SCHELL D, STEHLÍK M. The Harmonic Moment Tail Index Estimator: Asymptotic Distribution and Robustness [J]. Annals of the Institute of Statistical Mathematics, 2014, 66(1): 193-220.

[10] CAEIRO F, GOMES M I, BEIRLANT J, et al. Mean-of-Order p Reduced-Bias Extreme Value Index Estimation under a Third-Order Framework [J]. Extremes, 2016, 19(4): 561-589.

[11] BEIRLANT J, VYNCKIER P, TEUGELS J L. Tail Index Estimation, Pareto Quantile Plots Regression Diagnostics [J]. Journal of the American Statistical Association, 1996, 91(436): 1659-1667.

[12] RESINICK S I . Heavy-tail Phenomena [M]. New York: Springer, 2007.

[13] CASELLA G, BERGER R L. Statical Inference [M]. New Jersey: Prentice Hall, 2002.

[14] CRAMÉR H, WOLD H. Some Theorems on Distribution Functions [J]. Journal of the London Mathematical Society, 1936, s1-11(4): 290-294.

[15] PAULAUSKAS V, VAIČIULIS M. On an Improvement of Hill and some other Estimators [J]. Lithuanian Mathematical Journal, 2013, 53(3): 336-355.

责任编辑 张枸